Lecture 21

Review • Curve C parametrized by $\vec{r}(t) = (x(t), y(t))$ $a \le t \le b$
• function $f(x,y)$

* **Line integral of f along C** $\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$

To calculate: $\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex $\int_C xe^y\,ds$ C line segment from $(2,0)$ to $(5,4)$

$$C: \vec{r}(t) = (2,0) + t(3,4) = (3t+2, 4t) \quad 0 \le t \le 1$$

$$\int_0^1 (3t+2)e^{4t}\sqrt{9+16}\,dt = \int_0^1 15te^{4t} + 10e^{4t}\,dt$$
$$= -25/16 + \frac{85}{16}e^4$$

Rmk Replacing Δs_i by Δx_i or Δy_i we get

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$x = x(t)$
$dx = \frac{dx}{dt} \cdot dt$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Say $\int_C f(x,y)\,ds$ is line integral wrt arc length.

shorthand: $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$

Ex $\int_{\mathcal{C}} e^{y}\,dx$ $\quad C$ is arc of $y=x^3$ $(-1,1)$ to $(1,1)$

$\mathcal{C}$: $\vec{r}(t) = (t, t^3) \quad -1 \le t \le 1 \qquad dx = 1\,dt$

$$\int_{-1}^{1} e^{t}\,dt = e^{t}\Big|_{-1}^{1} = e - 1/e$$

Ex $\int_{\mathcal{C}} y\,dx + z\,dy + x\,dz$

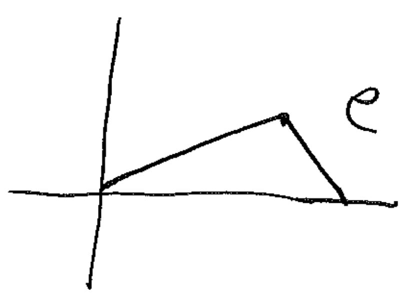
$\mathcal{C}$: $\vec{r}(t) = (\sqrt{t}, t, t^2) \quad 1 \le t \le 4$

$dx = \frac{1}{2\sqrt{t}}\,dt \quad dy = dt \quad dz = 2t\,dt$

$$= \int_1^4 t \cdot \frac{1}{2\sqrt{t}}\,dt + t^2\,dt + \sqrt{t}\cdot 2t\,dt$$

$$= \int_1^4 \frac{1}{2}\sqrt{t} + t^2 + 2t^{3/2}\,dt = 722/15$$

Ex $\int_{\mathcal{C}} (x+2y)\,dx + x^2\,dy$ $\qquad \mathcal{C}$ is line segment $(0,0)$ to $(2,1)$ then $(2,1)$ to $(3,0)$



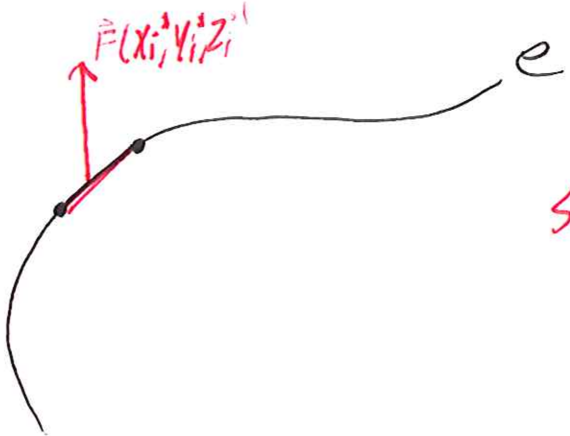
Prop Line integrals of piecewise smooth curves

Application Recall: Work = $\vec{F} \cdot \vec{D}$

Problem $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ force field (ex: gravity, electric, ...)

$\mathcal{C}$ space curve

What is total work done to move particle along $\mathcal{C}$ through $\vec{F}$?



small interval, displacement $\approx \vec{T}(\Delta s_i)$

unit tangent

$$\text{Work} \approx \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \Delta s_i$$

$\vec{T} = \dfrac{\vec{r}\,'(t)}{|r'(t)|}$ $\qquad$ $\Delta s_i \to |r'(t)| \cdot dt$

Conclude $$\text{Work} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Notation F continuous vector field defined on smooth curve $\mathcal{C}$ param by $\vec{r}(t)$ $a \le t \le b$. The line integral of F along $\mathcal{C}$ is

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\, dt$$

Rmk If $-\mathcal{C}$ is curve in opp direction then

$$\boxed{\int_{-\mathcal{C}} \vec{F}\cdot d\vec{r} = -\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}}$$

since unit tangent points opp

EX $\vec{F}(x,y) = xy^2\,\vec{i} - x^2\,\vec{j}$ $\quad \vec{r}(t) = t^3\,\vec{i} + t^2\,\vec{j}$ $\quad 0 \le t \le$

Evaluate $\int_{\mathcal{C}} F\cdot d\vec{r}$

EX Find work done by $\vec{F}(x,y) = (x^2, xy)$ on particle moving once around $x^2+y^2 = 4$ ccwise

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Observation Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ vector field

$$\mathcal{C}: \quad \vec{r}(t) = (x(t), y(t), z(t)) \quad a \le t \le b$$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(x(t),y(t),z(t)) \cdot (x'(t), y'(t), z'(t))\, dt$$

$$= \int_a^b P(x(t),y(t),z(t))\, x'(t) + Q(x(t),y(t),z(t))\, y'(t) + R(x(t),y(t),z(t))\, z'(t)\, dt$$

$$= \int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$$

Problems

Visualizer p. 1085 #17, 18

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47.