Lecture 20 - Vector Calculus

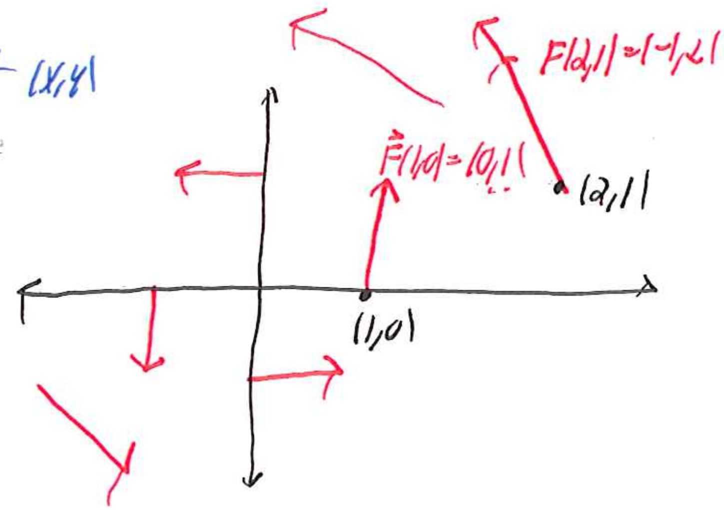
So far: Real valued $f: \mathbb{R}^n \to \mathbb{R}$ w/ exception of parametrized $\vec{r}(t): \mathbb{R} \to \mathbb{R}^n$.

Now More general $f: \mathbb{R}^n \to \mathbb{R}^m$.

Def Let $D \subseteq \mathbb{R}^2$. A vector field on D is a function $\vec{F}: D \to \mathbb{R}^2$.

Example $D = \mathbb{R}^2$; $F(x,y) = (-y, x)$. How to visualize?

* Draw arrow $\vec{F}(x,y)$ at (x,y)



Problems
- Calculate flow (flux..)
- Is vector field rotating?

Rmk Maple can quickly plot vector fields.

Rmk $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ also a vector field

Ex $\vec{F}(x,y,z) = \left(x\cos z,\ x+y^2,\ \frac{1}{z^2+1}\right)$

Ex Given $f: \mathbb{R}^n \to \mathbb{R}$, the gradient vector field is

$$\nabla f: \mathbb{R}^n \to \mathbb{R}^n$$

Ex $f(x,y,z,w) = w^2 + 2xyz \qquad \nabla f = (2yz, 2xz, 2xy, 2w)$

Vector fields arising in nature

Gravitation force: $\vec{F}(\vec{x}) = \dfrac{-mMG}{|\vec{x}|^3} \cdot \vec{x}$

$$\vec{F}(x,y,z) = \left(\frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(\quad)}, \frac{-mMGz}{(\,)} \right)$$

Ex Electric Fields

Ex Fluid Flow

Problems

1. Chpt 16 # 15-18, 29-32 matching.

2. Find ∇f

3. Particle moves in Velocity Field $\vec{V}(x,y) = \langle x^2, x+y^2 \rangle$ At time $t=3$ it is at $(2,1)$. Estimate location at $t=3.01$

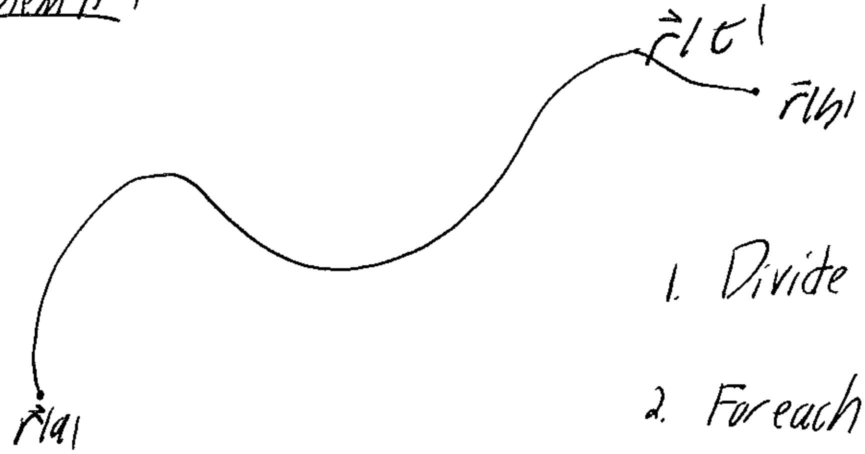
4. Sketch Flow Lines

Line integrals

Problem Curve $\vec{r}(t) = (x(t), y(t))$ in space, $a \le t \le b$

1. Given $f(x,y)$, integrate $f(x,y)$ along curve
2. Given vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$, integrate along curve.

Problem #1



1. Divide $[a,b]$ into pieces $[t_{i-1}, t_i]$
2. For each add up $f(x_i^*, y_i^*) \cdot$ Length
3. Length $\Delta s \approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t$

Conclude C a smooth curve given by $\vec{r}(t) = (x(t), y(t))$ $a \le t \le b$
f a function

Line integral of f along C is

$$\int_C f(x,y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Examples

1. $\int_C y\,ds \quad C: x=t^2, y=2t \quad 0 \le t \le 3$

2. $\cancel{\int_C x^2y+\sin} \quad \int_C xy^4\,ds \quad C:$ right half of $x^2+y^2=16$

Suppose Δs_i replaced by Δx_i or Δy_i. Get

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Call $\int_C f\,ds$ line integral w.r.t. arclength,