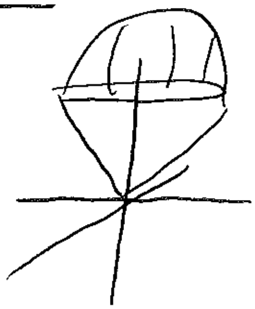
Lecture 19

Recall Spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta \qquad dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$z = \rho \cos\phi$$

EX Find volume above cone $z = \sqrt{x^2+y^2}$ below sphere $x^2+y^2+(z-1)^2 = 1$



Describe region: cone $\rho\cos\phi = \rho\sin\phi \Rightarrow \phi = \pi/4$

So $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$, $0 \le \rho \le \cos\phi$

Sphere $x^2+y^2+z^2 = z$

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\phi \sin\phi \, d\phi \, d\theta = \pi/8$$

EX Evaluate $\iiint_E \sqrt{x^2+y^2+z^2} \; dV$

E above cone $z = \sqrt{x^2+y^2}$

and betw spheres $x^2+y^2+z^2 = 4$

and $x^2+y^2+z^2 = 9$

in 1st octant.

EX $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)} \, dx\, dy\, dz = 2\pi$

Change of Variables

Calc I $\quad \int f(x)\,dx \quad x = g(u) \Rightarrow \int_a^b f(x)\,dx = \int_c^d f(g(u))\, g'(u)\, du$

$dx = g'(u)\,du$

$a = g(c)$

$b = g(d)$

One variable change of coord

Two variables

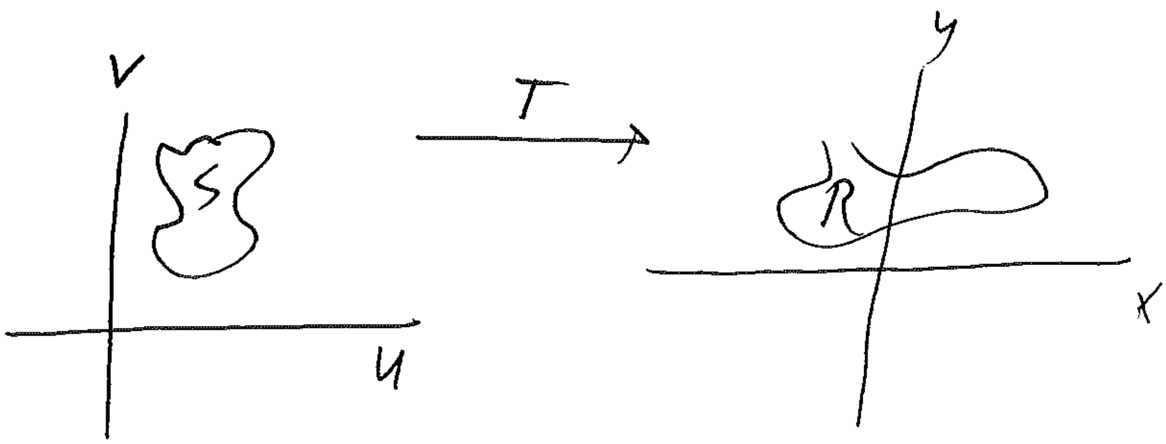
Suppose T: u-v plane $\Longrightarrow$ x-y plane

$T(u,v) = (x,y)$ where $x = g(u,v)$ $y = h(u,v)$

aka $x = x(u,v)$ $y = y(u,v)$

Ex $u = r$ $v = \theta$

$x = r\cos\theta$ $y = r\sin\theta$ $\qquad (r,\theta) \longrightarrow (x,y)$

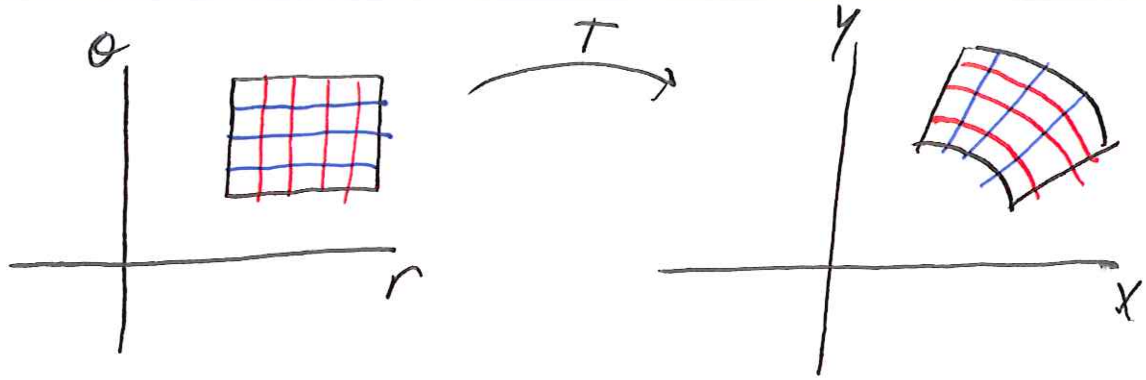


Assume 1. T is 1-1

2. g, h have continuous partial derivatives

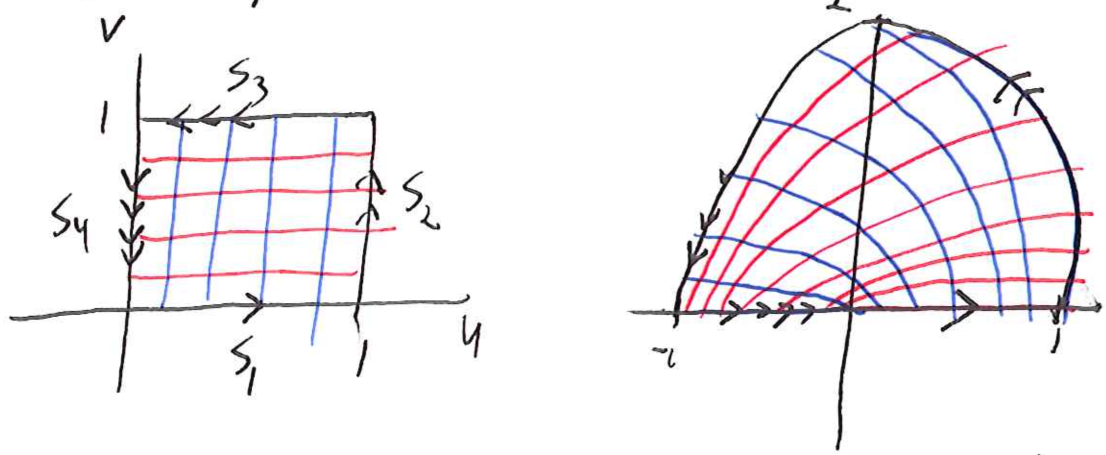
Rmk T has an inverse

Ex



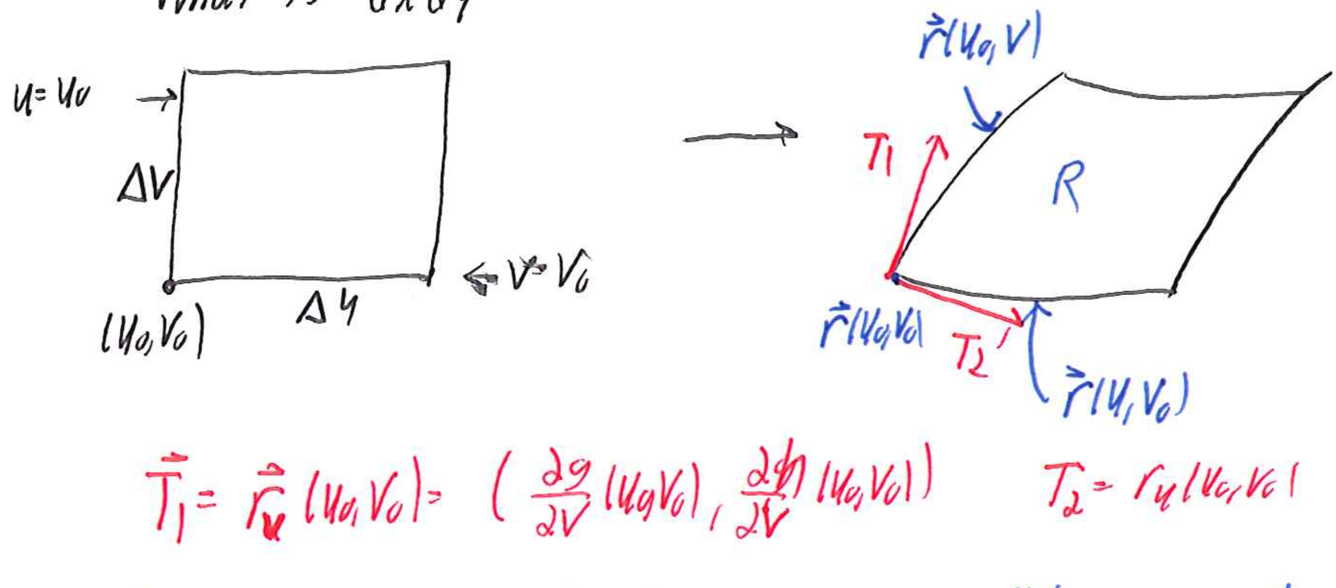
$T(r,\theta) = (r\cos\theta, r\sin\theta)$

Ex $\quad x = u^2 - v^2 \quad y = 2uv$



Q $\quad x = g(u,v) \quad y = h(u,v) \quad \vec{r}(u,v) = (g(u,v), h(u,v))$

What is $dx\,dy$?



$\vec{T}_1 = \vec{r}_v(u_0,v_0) = \left(\frac{\partial g}{\partial v}(u_0,v_0), \frac{\partial h}{\partial v}(u_0,v_0)\right) \qquad T_2 = r_u(u_0,v_0)$

* As $\Delta u, \Delta v$ are small, R is approx a parallelogram w/ sides $\vec{r}_v \Delta v$ & $\vec{r}_u \Delta u$

Area $R \approx |\vec{r}_u \Delta u \times \vec{r}_u \Delta v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

$$* \quad dx\,dy \Rightarrow |\quad| \, du\,dv$$

Def The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{dx}{du} & \frac{\partial x}{\partial v} \\ \frac{dy}{du} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Change of Variable Formula

Rmk Not easy to choose C.O.F

Rmk higher dim'l version