Lecture 14

Problem Find extreme values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$.

Lagrange Multiplier Method Idea At extrema, level curve of f is tangent to $g(x,y,z)=k$.

1. Find all (x,y,z,λ) so $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z)=k$.

   This is 3 equations

2. Test points

Ex Find extreme values of $f(x,y)=xy$ on ellipse $4x^2+y^2=8$

A: $\nabla f=(y,x)$ $\nabla g=(8x,2y)$

$$\left.\begin{aligned} y&=8\lambda x\\ x&=2\lambda y\\ 4x^2+y^2&=8 \end{aligned}\right\}$$ No general method, just ad-hoc.

$y=16\lambda^2 y \Rightarrow y(1-16\lambda^2)=0$ $y=0 \Rightarrow x=\pm\sqrt{2}$

$1-16\lambda^2=0 \Rightarrow \lambda=\pm 1/4$

$\lambda=1/4 \Rightarrow y=2x \Rightarrow 4x^2+4x^2=8 \Rightarrow x=\pm 1$ $(1,2)$ $(1,-2)$ $(-1,2)$ $(-1,-2)$

Point	xy
$(\sqrt{2},0)$	0
$(-\sqrt{2},0)$	0
$(1,2)$	2
$(1,-2)$	-2
$(-1,2)$	-2
$(-1,-2)$	2

max value 2 at $(1,2)$ & $(-1,-2)$
min value -2 at $(1,-2)$ & $(-1,2)$

Ex Find extreme values of $f(x,y,z)=e^{xyz}$ subject to

$$2x^2+y^2+z^2=2y$$

$\nabla f = (yze^{xyz}, xze^{xyz}, xye^{xyz})$ $\nabla g = (4x, 2y, 2z)$

$4x = \lambda yze^{xyz}$
$2y = \lambda xze^{xyz}$ $\Rightarrow$ $4x^2 = 2y^2 = 2z^2 = \lambda xyze^{xyz}$
$2z = \lambda xye^{xyz}$
$2x^2 + y^2 + z^2 = 24$ ← plug in $3z^2 = 24$ $z^2 = 8$ $z = \pm\sqrt{8}$

$4x^2 = 16$ $x = \pm 2$ $y = \pm\sqrt{8}$

8 points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$

max value e^{16} min value e^{-16}

Ex Find extreme value of $f(x,y) = x^2 + y^2 + 4x - 4y$ on $x^2 + y^2 \le 9$

$\nabla f = (2x+4, 2y-4)$ crit. point $(-2, 2)$

Now use Lagrange on boundary: $\nabla g = (2x, 2y)$

$2x + 4 = 2x\lambda$
$2y - 4 = 2y\lambda$ $\Rightarrow$ $2xy + 4y = 2xy - 4x$
$\Rightarrow x = -y$

$2x^2 = 9$ $x = \pm 3/\sqrt{2}$ $y = \pm 3/\sqrt{2}$

Test $(-2, 2)$, $(3/\sqrt{2}, -3/\sqrt{2})$, $(-3/\sqrt{2}, 3/\sqrt{2})$

Two constraints Max/min $f(x,y,z)$ subj to $g(x,y,z) = k$,
$h(x,y,z) = c$

- ∇f is in plane spanned by ∇g & ∇h.

Solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$
$g(x,y,z) = k$ 5 equations
$h(x,y,z) = c$ 5 unknowns x, y, z, λ, μ

EX Find extreme values of $f(x,y,z) = yz + xy$ s.t.
$xy = 1 \quad y^2 + z^2 = 1$

A $\nabla f = (y, z+x, y) \quad \nabla g = (y, x, 0) \quad \nabla h = (0, 2y, 2z)$

$y = \lambda y + 0$ $\Rightarrow y = 0$ ruled out so $\lambda = 1$
$z + x = \lambda x + 2\mu y$ $\Rightarrow$ Thus $z = 2\mu y = 4\mu^2 z$
$y = 2\mu z$
$xy = 1$
$y^2 + z^2 = 1$

$z = 4\mu^2 z \Rightarrow$ $z = 0$ or $\mu = \pm 1/2$

$z = 0 \Rightarrow y = 0$ $\cancel{*}$
Thus $\lambda = 1, \mu = \pm 1/2 \quad y = \pm z$, $x = 1/y$ plug into $y^2 + z^2 = 1$

End of Chpt 14

Multiple integrals

Review $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

Problem Suppose $f(x,y) \geq 0$. Find volume above $[a,b] \times [c,d]$ and below graph of f.

- Use same idea
  - rectangles, points (x_{ij}^*, y_{ij}^*), ΔA

Def The double integral of $f(x,y)$ over a rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

double Riemann sum

Ex $R = [0,4] \times [0,2]$ $f = 25 - x^2 - y^2$ estimate w/ 4 rectangles, center of each

* Pictures from Stewart

Iterated integrals

$$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

hold x constant, this is area $A(x)$

Ex $\int_0^4 \int_1^3 xy^3\,dy\,dx$ $\int_1^3 \int_0^4 xy^3\,dx\,dy$

Fubini Thm Suppose $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$

Then $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

$\boxed{\text{Animation}}$

Ex $\iint \frac{xy^2}{x^2+1}\,dA$ $0 \le x \le 1$ $-3 \le y \le 3$