Lecture 13

Review $f(x,y,z,\ldots)$ function, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \ldots\right)$ gradient vector

- $D_{\vec{u}} f(x,y,\ldots) = \nabla f \cdot \vec{u}$, max rate of change at point $(x_0, y_0, \ldots)$ is $|\nabla f(x_0, y_0, \ldots)|$ in direction ∇f.

Thm If f has a local max/min at point then either $\underbrace{\nabla f = 0 \text{ or DNE}}_{\text{critical point}}$.

* Review def of local max/min vs global max/min

Second Der. Test Suppose 2^{nd} partials are cont. on disc containing (a,b).
Suppose $\nabla f(a,b) = (0,0)$

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2.$$

a. $D > 0$, $f_{xx}(a,b) > 0 \Rightarrow$ local min
b. $D > 0$, $f_{xx}(a,b) < 0 \Rightarrow$ local max
c. $D < 0 \Rightarrow$ saddle point

$D = 0$ no info

14.7 #13 $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

$\nabla f = (4x^3 - 4x,\ 3y^2 - 3)$ set $\nabla f = (0,0)$ get $x = 0, \pm 1$
$y = \pm 1$

six critical points! $(0,1)$ $(0,-1)$ $(1,1)$ $(1,-1)$ $(-1,1)$ $(-1,-1)$

$f_{xx} = 12x^2 - 4$ $f_{yy} = 6y$ $f_{xy} = 0$

$$D = (12x^2 - 4)(6y)$$

Classify crit points

pt	D	f_{xx}	
$(0,1)$	-24		saddle
$(0,-1)$	24	-4	max
$(1,1)$	48	8	min
$(1,-1)$	-48		saddle
$(-1,1)$	48	8	min
$(-1,-1)$	-48		saddle

local max $f(0,-1) = 2$

local min $f(1,1) = -3$

local min $f(-1,1) = -3$

Ex $f(x,y) = \sin x \sin y \quad -\pi < x < \pi \quad -\pi < y < \pi$

$\nabla f = (\cos x \sin y, \sin x \cos y)$ set $= (0,0)$ 5 points $(0,0)$, $(\pm \pi/2, \pm \pi/2)$

$f_{xx} = -\sin x \sin y \quad f_{yy} = -\sin x \sin y \quad f_{xy} = f_{yx} = \cos x \cos y$

$$D = +\sin^2 x \sin^2 y - (\cos x \cos y)^2$$

	D	f_{xx}	
$(0,0)$	-1		saddle
$(\pi/2, \pi/2)$	$+1$	-1	~~saddle~~ max
$(\pi/2, -\pi/2)$	$+1$	1	min
$(-\pi/2, \pi/2)$	$+1$	1	min
$(-\pi/2, -\pi/2)$	1	-1	max

local max $f(\pi/2, \pi/2) = 1$

$f(-\pi/2, -\pi/2) = 1$

local min $f(\pi/2, -\pi/2) = -1$

$f(-\pi/2, \pi/2) = -1$

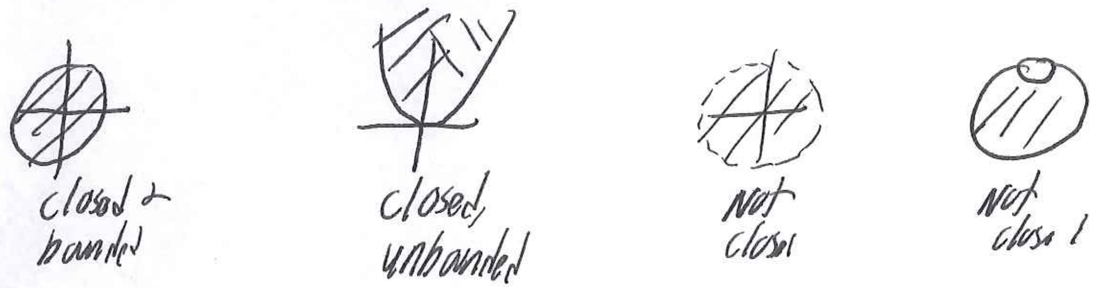
<u>Review</u> $f(x)$ continuous on $[a,b]$, then $f(x)$ attains a global max & min on $[a,b]$ and they occur at

- crit points
- endpoints

<u>Multivariable</u> "Closed set" contains all boundary pts

"bounded set" contained in some disk about (0,0)

<u>Ex</u>



<u>Thm</u> Let $f(x,y)$ be continuous on a closed, bounded set $D \subseteq \mathbb{R}^2$. Then f attains an absolute max value $f(x_1,y_1)$ & absolute min value $f(x_2,y_2)$ for $(x_1,y_1), (x_2,y_2) \in D$

Moreover they occur at critical pts or on boundary.

* Checking boundary usually reduces to a calc I problem

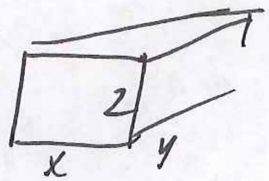
<u>EX</u> 14.7 # 33 $f(x,y) = x^2+y^2+x^2y+y$ on

$$D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$$

EX $f(x,y) = 2x^3 + y^4$ on $D = \{x^2+y^2 \le 1\}$

Ex 14.7 Ex 6

Rectangular box, no lid, $12m^2$ of cardboard. Maximize volume



$V = xyz$

Give $xy + 2xz + 2yz = 12$

solve $z = \dfrac{12 - xy}{2x + 2y}$

$$V(x,y) = xy \, \frac{12 - xy}{2(x+y)}$$

$$V_x = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad V_y = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2} \rightsquigarrow x = y$$

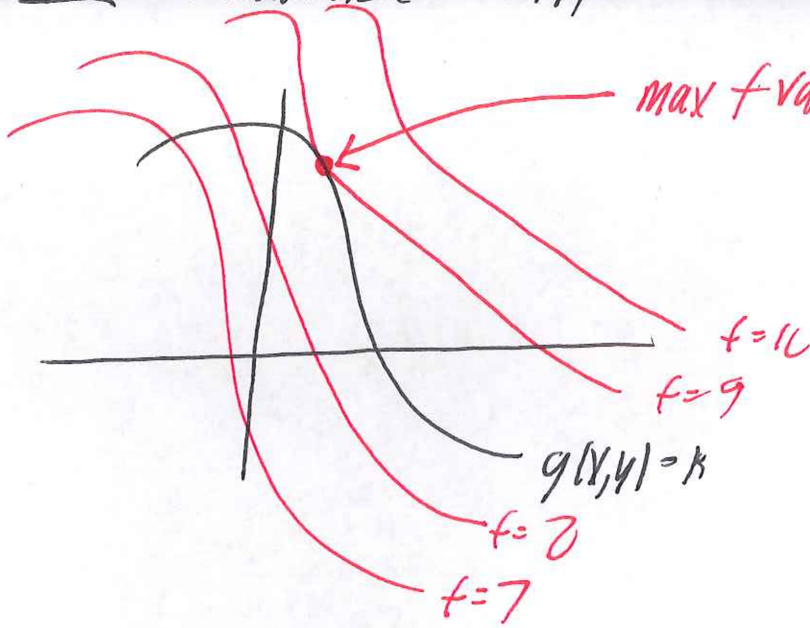
Get $x = 2$ $y = 2$ $z = 1$ //

Rmk This is an example of optimizing a function

$V(x,y,z) = xyz$ subject to a constraint $xy + 2xz + 2yz = 12$

Goal! Systematically solve such

Idea Maximize $f(x,y)$ subject to $g(x,y)=k$



* seems that max value comes at a point where level curve of f is // to $g(x,y)=k$.

$$\Longleftrightarrow \nabla f \parallel \nabla g$$

Lagrange Multiplier

To find max/min values of $f(x,y,z)$ sub to $g(x,y,z)=k$, assuming they exist and $\nabla g \neq 0$ on $g=k$. Then

1. Solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ (3 eqs)

$$g(x,y,z)=k$$

2. Test points

Ex Redo Box