Lecture 12

Problem: Given $z = f(x,y)$, what is R.O.C. of $f(x,y)$ in a certain direction?

Def (x_0, y_0) a point, $\vec{u} = (a,b)$ a unit vector. Then

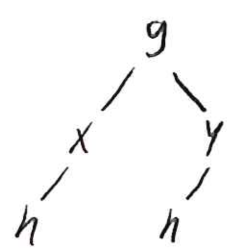
$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

is the <u>directional derivative</u>, of f at (x_0, y_0) in direction (a,b)!

Ex Partial derivatives are case $\vec{u} = (1,0)$ and $\vec{u} = (0,1)$!

How to compute?

Let $g(h) = f(x_0 + ha, y_0 + hb)$ $\quad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = D_{\vec{u}} f$



$$g'(h) = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} b \qquad (*)$$

where $x = x_0 + ha$
$y = y_0 + hb$

Def Given $f(x,y)$, the <u>gradient vector</u> is $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$

Ex $f(x,y) = x^2 y + y \quad \nabla f = (2xy, x^2 + 1)$

$f(x,y,z) = xyz \quad \nabla f = (yz, xz, xy)$

Thm f diffble function of x, y. Then f has a dir. derivative in any direction $\vec{u} = (a,b)$ and

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$$

Ex $f(x,y) = xy^3 - x^2$. Find dir der at $(1,2)$ in dir. $(1,1)$

$$\nabla f = (y^3 - 2x,\ 3xy^2) \qquad \vec{u} = (1/\sqrt{2}, 1/\sqrt{2})$$

$$\nabla f(1,2) = (6, 12) \qquad (6,12)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 18/\sqrt{2}$$

Ex $f(x,y,z) = xy^2 \tan^{-1} z$ Find dir der at $(2,1,1)$ in dir $(1,2,3)$

$$\nabla f = \left(y^2\tan^{-1}z,\ 2xy\tan^{-1}z,\ \frac{xy^2}{1+z^2}\right) \qquad \vec{u} = (1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$$

$$\nabla f(2,1,1) = (\pi/4, \pi, 1)$$

$$(\pi/4, \pi, 1)\cdot\vec{u} = \frac{9\pi/4 + 3}{\sqrt{14}}$$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||u|\cos\theta$ $\Rightarrow$ max value when $\vec{u}$ parallel to ∇f

Thm. The maximum value of $D_{\vec{u}} f(x,y)$ is $|\nabla f(x,y)|$ and it occurs then $\vec{u}$ has the same direction as the gradient vector.

* Show maple worksheet.

Problem $f(x,y) = \sin(xy)$. Find max rate of change and direction it occurs for at point $(1,0)$

$$\nabla f = (y\cos(xy), x\cos(xy)) \qquad \nabla f(1,0) = (0,1)$$

max rate $= \|(0,1)\| = 1$ dir $(0,1)$

Problem Repeat for $f(x,y,z) = x\ln(yz)$

Key Observation: ∇f is $\perp$ to level surfaces $f=k$

Proof: Let $F(x,y,z)=k$ a level surface, let $\vec{r}(t)=(x(t),y(t),z(t))$ a curve on surface. Then

$$F(x(t),y(t),z(t))=k \qquad \text{apply } \frac{d}{dt}:$$

$$\frac{\partial F}{\partial x}x'(t)+\frac{\partial F}{\partial y}y'(t)+\frac{\partial F}{\partial z}z'(t)=0 \qquad \nabla F\cdot\vec{r}\,'(t)=0.$$

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Cor: Can Find Tangent plane to level surface by using ∇F as normal:

Example: $z=x^2+2y^2$ at $(2,3,22)$

$F(x,y,z)=x^2+2y^2-z$ Level curve $F(x,y,z)=0$

$\nabla F=(2x,4y,-1)$ $\nabla F(2,3,22)=(4,12,-1)$

$$\boxed{4x+12y-z=22}$$

Ex: Find eq of tangent plane and normal line to

$$xy+yz+zx=5 \text{ at } (1,2,1)$$

A: $F(x,y,z)=xy+yz+zx-5$

$\nabla F\,(y+z,x+z,y+x)$ $\nabla F(1,2,1)=(3,2,3)$

P: $3x+2y+3z=10$

line: $\vec{r}(t)=(1,2,1)+t(3,2,3)$

Summary

- $\nabla f \cdot \vec{u}$ gives directional deriv. $D_{\vec{u}} f$
- rate of maximum increase is $|\nabla f(x,y)|$ in direction $\nabla f(x,y)$
  " " " decrease is $-|\nabla f(x,y)|$ " " $-\nabla f$
- Level surface $F(x_1, x_2, \ldots, x_n) = 0$ then ∇F is $\perp$ to level surface.

Ex Show that $3x^2+2y^2+z^2=9$ and $x^2+y^2+z^2-8x-6y-8z+24$ are tangent at $(1,1,2)$

Optimization

Rmk If $f(x,y)$ has a local extrema, then $\nabla f(x,y)$ should be $(0,0)$ or DNE.

Define Local & Global max min

Thm If $f(x,y)$ has local max/min at (a,b) and f_x, f_y exist at (a,b) then
$$f_x(a,b) = 0 = f_y(a,b)!$$

Def (a,b) is a critical pt if $\nabla f(a,b) = (0,0)$ or DNE

Ex Show $f(x,y) = y^2 - x^2$ has no local max/min

Second Der Test

Suppose f_x, f_y continuous on disk containing (a,b) and $\nabla f(a,b) = (0,0)$

Let $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$

a. $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ a local min

b. $D > 0$ and $f_{xx}(a,b) < 0$ then local max

c. $D < 0$ saddle (neither)

d. $D = 0$ no info