Lecture 11

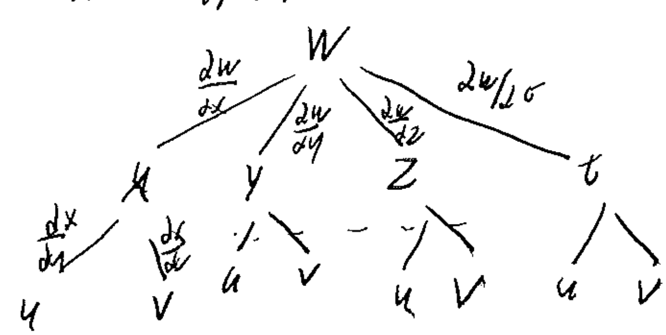
Recall:

<u>Chain Rule</u> Suppose u is a diffble function of variables $x_1, x_2, \dots, x_n$ and each x_i is a diffble function of $t_1, t_2, \dots, t_m$. Then

for each i:
$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

<u>Ex</u> $w = f(x, y, z, t) \quad x = x(u,v) \quad y = y(u,v) \quad z = z(u,v) \quad t = t(u,v)$

tree diagram:



<u>Ex</u>
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

<u>Ex</u> $z = (x-y)^5 \quad x = s^2 t \quad y = st^2$ Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

<u>Ex</u> $w = xy + yz + zx \quad x = r\cos\theta \quad y = r\sin\theta \quad z = r\theta$

Find $\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta}$ when $r = 2, \ \theta = \pi/2$

Ex Suppose $f(x,y)$ is diffble and $g(u,v) = f(e^u + \sin v, e^u + \cos v)$.

Find $g_u(0,0)$ & $g_v(0,0)$

	f	g	f_x	f_y
$(0,0)$	3	6	4	8
$(1,2)$	6	3	2	5

Ex $z = \tan(u/v)$ $u = 2s + 3t$, $v = 3s - 2t$. Find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$

Implicit diff from calc 1

Ex $x^2 y + \cos(xy) = 3$ Find $\frac{dy}{dx}$

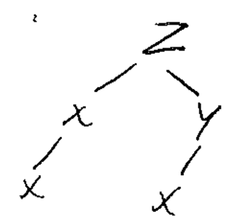
A: $2xy + x^2 y' - \sin(xy)\,[xy' + y] = 0$

$$x^2 y' - x\sin(xy)\,y' = -2xy + y\sin(xy)$$

$$y' = \frac{-2xy + y\sin(x,y)}{x^2 - x\sin(xy)}$$

Alternate approach

Given $F(x,y) = 0$. Apply $\frac{\partial}{\partial x}$



$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$\rightsquigarrow \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Conclude: Given $F(x,y)=0$, $\boxed{\dfrac{\partial y}{\partial x} = \dfrac{-F_x}{F_y}}$

Back to our example: $F(x,y) = x^2y + \cos(xy) - 3$

$F_x = 2xy - y\sin(x,y)$

$F_y = x^2 - x\sin(xy)$

• Rmk on Implicit Function Thm

<u>Generalize</u> Given $F(x,y,z)=0$, consider z an implicit function of (x,y) Then:

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

<u>Ex</u> (Exam #7) $x^2z + y^2 + z^2 = 3$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-2xz}{x^2+2z}$$

<u>Ex</u> $e^z = xyz$ Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

Ex Suppose $z=f(x,y)$ has continuous second-order partials and $x=r^2+s^2$, $y=2rs$

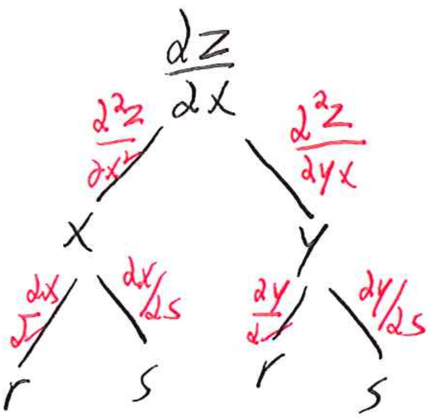
Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$

A: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = 2r \cdot \dfrac{\partial z}{\partial x} + 2s\dfrac{\partial z}{\partial y}$

Apply $\dfrac{\partial}{\partial r}$:

$$\frac{\partial^2 z}{\partial r^2} = 2\cdot \frac{\partial z}{\partial x} + 2r\left(\underbrace{\frac{\partial}{\partial r}\frac{\partial z}{\partial x}}\right) + 2s\underbrace{\frac{\partial}{\partial r}\frac{\partial z}{\partial y}}$$

use chain rule !



$$= 2\frac{\partial z}{\partial x} + 2r\left(\frac{\partial^2 z}{\partial x^2}\cdot 2r + \frac{\partial^2 z}{\partial y x}2s\right) + 2s\left(\frac{\partial^2 z}{\partial x \partial y}\cdot 2r + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= \boxed{2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 y}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}}$$

#49 Show any function $z = f(x+at) + g(x-at)$ is a solution of wave eq: $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

#40 in 14.5