Lecture 10

- Review exam

Review Graph $z = f(x,y)$. Point (x_0, y_0, z_0) on graph, so $z_0 = f(x_0, y_0)$.

* The tangent plane at (x_0, y_0, z_0) is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Ex Find eq of tangent plane to $z = x/y^2$ at $(-4, 2, -1)$

$$f_x = 1/y^2 \qquad f_y = -2xy^{-3} = \frac{-2x}{y^3}$$

$$f_x(-4,2) = 1/4 \qquad f_y(-4,2) = 1 \qquad f(-4,2) = -1$$

$$\boxed{z + 1 = \frac{1}{4}(x+4) + 1(y-2)}$$

Linear approx

Idea: The tangent plane

$$z = \underbrace{\boxed{z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}}_{L(x,y)}$$

is a good approximation to $z = f(x,y)$ near (x_0, y_0)

Linear approximation

$$f(x,y) = f(a,b) + \underbrace{f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{\text{call this the differential } dz}$$

so $f(x,y) \approx f(a,b) + dz$

So $dz = f_x(x,y)\,dx + f_y(x,y)\,dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$

Problems

1. $f(x,y) = \frac{1+y}{1+x}$. Find linear approx near $(1,3)$

2. $Z = e^{-2x}\cos t$. Find dz.

$$dz = -2e^{2x}\cos t\,dx - e^{-2x}\sin t\,dt$$

3. $Z = 5x^2+y^2$ (x,y) changes from $(1,2)$ to $(1.05, 2.1)$

Compare values of ΔZ and dz

Note $f(1,2) = 9$ $f(1.05, 2.1) = 1.105 \cdot 5 + 4.41 = 9.935$

Chain Rule

Review Suppose $y = f(x)$ and $x = g(t)$ so $y = f(g(t))$

Then $y' = f'(g(t)) \cdot g'(t) = \frac{df}{dx}(g(t)) \cdot \frac{dg(t)}{dt}$

$$= \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Idea Generalize to $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^s$

so $g(f(\vec{v}))$ maps $\mathbb{R}^n \to \mathbb{R}^s$, many partial derivatives!

Special Case 1 $z = f(x,y)$ w/ $x = g(t)$, $y = h(t)$

$$t \longrightarrow (x,y) \longrightarrow z$$

Thm Suppose $z = f(x,y)$ is differentiable where $x = g(t)$ & $y = h(t)$ are diff'ble functions of t

Then z is a diff'ble function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

also write $\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Ex

1 $Z = \sin x \cos y \quad x = \sqrt{t} \quad y = 1/t \quad$ Find $\frac{dz}{dt}$

2 $Z = (x-y)^5 \quad x = e^t \quad y = t^2$

Case 2 Now suppose $x = g(s,t)$ $y = h(s,t)$ are diff'ble

and $Z = f(x,y)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex $Z = e^r \cos\theta \quad r = st \quad \theta = \sqrt{s^2+t^2}$

Next General case

u a function of $(x_1, x_2, , x_n)$

Each x_i a function of $t_1, t_2, , t_m$

$$\frac{\partial u}{\partial t_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial t_i}$$