Math 241S- Final Examination- December 1~~5~~7, 2015

NAME: SOLUTIONS

| Problem # | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 15 | |
| 10 | 10 | |
| 11 | 10 | |
| 12 | 15 | |
| 13 | 10 | |
| 14 | 10 | |
| **Total** | 150 | |

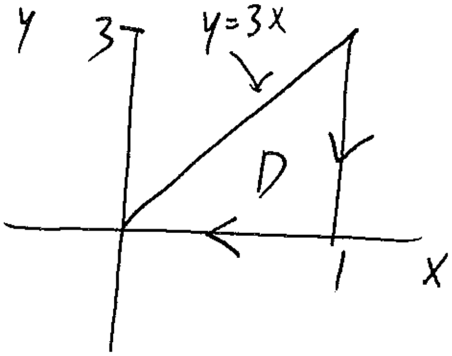
~~Math 241S Midterm Exam #3 - December 10, 2015~~

**Instructions:** You may have notes covering no more than a single side of an 8x11 piece of paper. No other aides, electronic or otherwise, are allowed.

1. **(10 points)** Use Green's Theorem to evaluate:

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

where $C$ is the triangle with vertices $(0,0), (1,3), (1,0)$ oriented *clockwise.*



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 0 = 2y$$

$$-\int_0^1 \int_0^{3x} 2y\,dy\,dx = -\int_0^1 y^2\Big|_0^{3x} dx$$

since clockwise

$$= -\int_0^1 9x^2\,dx$$

$$= 3x^3\Big|_0^1$$

$$= \boxed{-3}$$

2. **(10 points)** Find a unit vector perpendicular to the plane containing the 3 points $(1,0,0),(0,2,0),(0,0,3)$. What is the area of the triangle with these 3 vertices?

A B C

$\vec{AB} = (-1,2,0)$

$\vec{AC} = (-1,0,3)$

$\vec{AB} \times \vec{AC} = (6,3,2)$ has magnitude $\sqrt{36+9+4} = 7$

$$\boxed{(6/7, 3/7, 2/7)}$$

area $\Delta = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \boxed{7/2}$

3. **(10 points)**

a. Find an equation for the plane passing through $\overset{P}{(3,-1,1)}, \overset{Q}{(4,0,2)}$ and $\overset{R}{(6,3,1)}$.

b. Given the points $A(1,0,1)$, $B(2,3,0)$, $C(-1,1,4)$, and $D(0,3,2)$, find the volume of the parallelepiped with adjacent edges $AB, AC$ and $AD$.

a. $\vec{PQ} = (1,1,1) \quad \vec{PR} = (3,4,0)$

$$\vec{PQ} \times \vec{PR} = (-4,3,1)$$

$$\boxed{\begin{gathered} (-4,3,1)\cdot(x-3, y+1, z-1) = 0 \\ \text{or} \\ -4x+3y+z = -14 \end{gathered}}$$

b. $\vec{AB} = (1,3,-1)$

$\vec{AC} = (-2,1,3)$

$\vec{AD} = (-1,3,1)$

$$V = |\vec{AB}\cdot(\vec{AC}\times\vec{AD})|$$

$$= |(1,3,-1)\cdot(-8,-1,-5)|$$

$$= |-6|$$

$$= \boxed{6}$$

4. **(10 points)** a. A particle moves with position function $\mathbf{r}(t) = (t\ln t, t, e^{-t})$. Find the velocity, speed and acceleration of the particle.

b. Consider a curve with parametric equations $x = t^2 + 1$, $y = 4\sqrt{t}$, $z = e^{t^2-t}$. Find the parametric equation for the tangent line to this curve at the point $(2, 4, 1)$.

a. $\vec{v}(t) = (\ln t + 1, 1, -e^{-t})$

speed $= |\vec{v}(t)| = \sqrt{(\ln t + 1)^2 + 1 + e^{-2t}}$

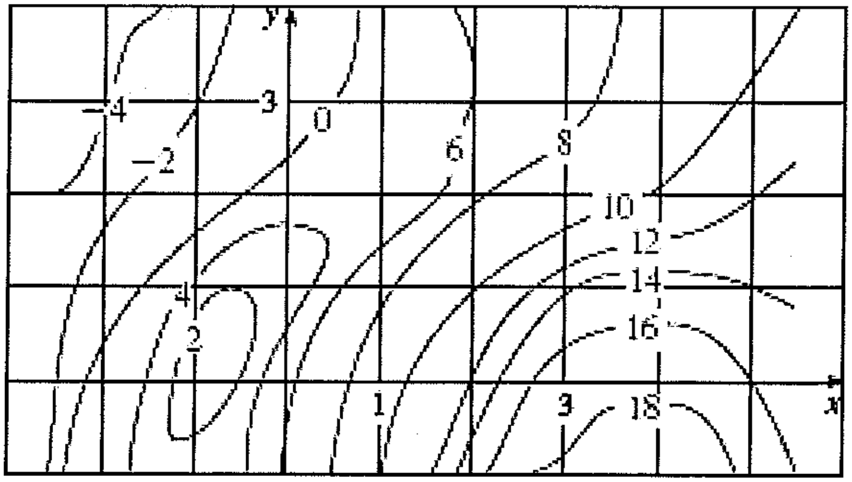
$\vec{a}(t) = (1/t, 0, e^{-t})$

b $\vec{r}'(t) = (2t, 2/\sqrt{t}, (2t-1)e^{t^2-t})$

$t = 1$

$\vec{r}'(1) = (2, 2, 3)$

$$\boxed{(2,4,1) + t(2,2,3)}$$



5. **(10 points)**
Above is a contour plot for a function $f(x, y)$.
a. Estimate the partial derivatives $f_x(2, 1)$ and $f_y(2, 1)$. Show your work.

b. Is $f_{yy}(3, 2)$ positive or negative? Explain.

See Midterm1 #9

6. **(10 points)** Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$. In what direction does it occur?

$$\nabla f = (2xy,\ x^2 + 1/2\sqrt{y})$$

$$\nabla f(2,1) = (4,\ 9/2) \leftarrow \text{Direction}$$

$$\text{max rate} = |\nabla f(2,1)| = \sqrt{16 + 81/4}$$

$$= \sqrt{145/4}$$

$$\boxed{= \sqrt{145}/2}$$

7. **(10 points)** Suppose

$$\cos(xz) = 1 + x^2y^2 + z^2.$$

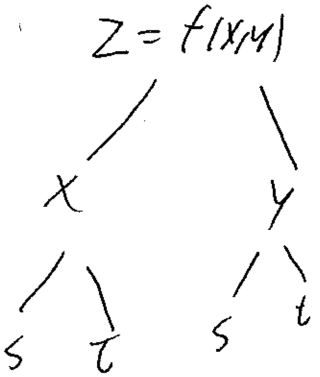
Find $\frac{\partial z}{\partial x}$.

$$-\sin(xz)(z + x\frac{\partial z}{\partial x}) = 0 + 2xy + 2z\frac{\partial z}{\partial x}$$

$$-z\sin(xz) - x\sin(xz)\frac{dz}{dx} = 2xy + 2z\frac{dz}{dx}$$

$$\boxed{\frac{dz}{dx} = \frac{-z\sin(xz) - 2xy}{x\sin(xz) + 2z}}$$

8. **(10 points)** Suppose $z = f(x, y)$ where $x = g(s, t), y = h(s, t)$. Suppose further that $g(1, 2) = 3, g_s(1, 2) = -1, g_t(1, 2) = 4, h(1, 2) = 6, h_s(1, 2) = -5, h_t(1, 2) = 10, f_x(3, 6) = 7$ and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.



$s=1 \quad t=2 \Rightarrow x=3$

$y=6$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$= f_x(3,6)\, g_s(1,2) + f_y(3,6)\, h_s(1,2)$$

$$= 7\cdot(-1) + 8\cdot(-5) = \boxed{-47}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= 7(4) + 8(10) = \boxed{108}$$

9. **(15 points)** Let $f(x, y) = 2x^3 - 6xy + 3y^2$. Find all local maximum and minimum values and saddle points of $f(x, y)$.

$\nabla f = (6x^2 - 6y, -6x + 6y)$

Set $= (0,0)$ to get $y = x^2 \quad y = x$

$\Rightarrow x = x^2 \Rightarrow x = 0$ or $1$

Crit points $(0,0)$, $(1,1)$

$f_{xx} = 12x \qquad f_{yy} = 6 \qquad f_{xy} = -6$

$D = 72x \overset{-}{\cancel{+}} 36$

| Pt | D | $f_{xx}$ | |
|---|---|---|---|
| $(0,0)$ | $-36$ | N/A | ← saddle |
| $(1,1)$ | $36$ | $72$ | ← local min |

10. **(10 points)** Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

$\nabla f = (y, x) \quad \nabla g = (8x, 2y)$

$y = 8\lambda x \quad \Rightarrow \quad y = 16\lambda^2 y$
$x = 2\lambda y$
$4x^2 + y^2 = 8$

If $y=0$ then $x=0$ $\#$ So $y \neq 0$ and $16\lambda^2 = 1$

$\lambda = \pm 1/4$

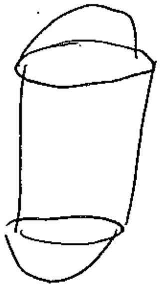
$\lambda = 1/4 \Rightarrow y = 2x \Rightarrow 8x^2 = 8 \Rightarrow x = \pm 1$
$(1, 2)$
$(-1, -2)$

$\lambda = -1/4 \Rightarrow y = -2x \Rightarrow 8x^2 = 8 \Rightarrow x = \pm 1$
$(1, -2)$
$(-1, 2)$

| pt | f |
|---|---|
| $(1,2)$ | 2 |
| $(-1,-2)$ | 2 |
| $(1,-2)$ | $-2$ |
| $(-1,2)$ | $-2$ |

max value 2
min value $-2$

11. **(10 points)** Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.



cylindrical: $0 \le r \le 1$

$0 \le \theta \le 2\pi$

$-\sqrt{4-r^2} \le z \le \sqrt{4-r^2}$

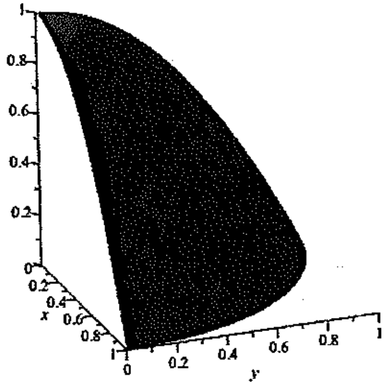
$$V = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 2r\sqrt{4-r^2} \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{-2}{3}(4-r^2)^{3/2} \Big|_0^1 \, d\theta$$

$$= \boxed{\frac{-4\pi}{3}\left(3^{3/2} - 4^{3/2}\right)}$$

12. **(15 points)** Consider the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant,



shown above:

Let $C$ be the boundary of this surface, oriented counterclockwise when viewed from above. Let:

$$\mathbf{F}(x, y, z) = (zy, yz, zx).$$

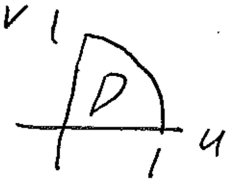
Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

+ Set up only

$\text{curl}\,\vec{F} = ( -y, y - z, -z )$

$\vec{r}(u,v) = (u, v, 1-u^2-v^2)$

$(u,v) \in$



$r_u = (1, 0, -2u)$

$r_v = (0, 1, -2v)$

$r_u \times r_v = (2u, 2v, 1)$

$$\iint_D \text{curl}\,F(r(u,v)) \cdot r_u \times r_v = \iint_D \overset{-2v}{\cancel{(2u-1,}}\ 2v-1, -1) \cdot (2u, 2v, 1)$$

$$= \iint_D (-v, v-1+u^2+v^2, u^2+v^2-1) \cdot (2u, 2v, 1)\, du\, dv$$

or in polar $D$: $0 \le \theta \le \pi/2$

$0 \le r \le 1$

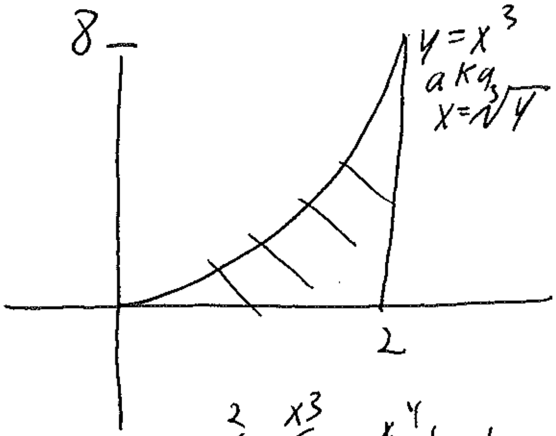
$u = r\cos\theta$

$v = r\sin\theta$

$du\,dv = r\,dr\,d\theta$

13. **(10 points)** Evaluate the integral by reversing the order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy.$$



$0 \leq y \leq x^3$

$0 \leq x \leq 2$

$$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4} e^{x^4}\Big|_0^2$$

$$= \frac{e^{16}-1}{4}$$

14. **(10 points)** Let

$$\mathbf{F}(x, y, z) = (\sin y, x\cos y + \cos z, -y\sin z).$$

a. Show $\mathbf{F}$ is conservative by finding a potential function.

b. Let $C$ be the curve $\mathbf{r}(t) = (\sin t, t, 2t) 0 \le t \le \pi/2$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem of line integrals.

a. $f = X\sin y + y\cos z$ has $\nabla f = \vec{F}$

b $f(\vec{r}(\pi/2)) = f(1, \pi/2, \pi) = 1 - \pi/2$

$f(\vec{r}(0)) = f(0,0,0) = 0$

By F.T. $\int \nabla f d\vec{r} = f(r(b)) - f(r(a))$

$$\boxed{= 1 - \pi/2}$$