NAME: SOLUTIONS

Problem #	Points	Score
1	25	
2	10	
3	10	
4	10	
5	15	
6	15	
7	15	
Total	100	

Math 241S- Midterm Exam #3 - December 10, 2015

Instructions: You may not use any outside help, including notes, index cards, electronic devices or your neighbor!

1. **(25 points)** Short answer, very limited partial credit.

a. Define: A vector field $\mathbf{F}$ is *conservative* if $\cdots$

$\vec{F} = \nabla f$ for some function f.

b. Sketch a region in the plane which is connected but is not simply connected.

c. Let $\mathbf{F}(x, y, z) = (y + 2xz, x + z, y + x^2)$. Find a potential function for $\mathbf{F}$.

$f_x = y + 2xz$ so $f = xy + x^2 z + C(y,z)$

$f_y = x + z$ so $f = xy + yz + C(x,z)$

$f_z = y + x^2$ so $f = x^2 z + yz + C(x,y)$

$$\boxed{f(x,y,z) = xy + yz + x^2 z}$$

d. Define: A vector field $\mathbf{F}$ is *irrotational* if $\cdots$

$\text{curl} F = \vec{0}$

e. State the Fundamental Theorem for Line Integrals.

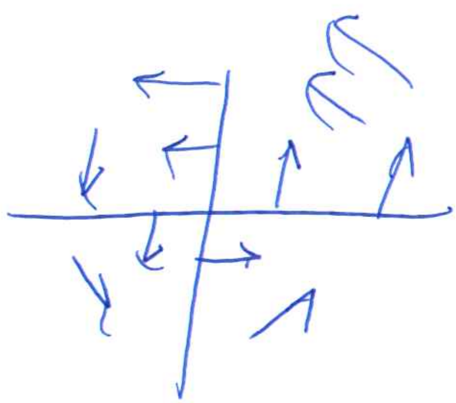
Let C be a smooth curve given by $\vec{r}(t)$ $a \leq t \leq b$

Suppose ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

f. Sketch a vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ on $\mathbb{R}^2$ such that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

Any field w/ obvious rotation



2. **(10 points)** Consider the surface parameterized by $r(s,t) = (t, s^2 t, t^2 + 3s)$. Find the equation of the tangent plane to this surface at the point $(2,2,7)$. Give your answer in two forms. First parameterize the equation of the plane. Then put your answer in the form $ax + by + cz = d$.

$(2,2,7)$ is $s=1$ $t=2$

$r_s = (0, 2st, 3)$ $\quad r_s(1,2) = (0,4,3)$

$r_t = (1, s^2, 2t)$ $\quad r_t(1,2) = (1,1,4)$

$r_s \times r_t = (13, 3, -4)$

$$\boxed{\begin{gathered}(2,2,7) + s(0,4,3) + t(1,1,4) \\ 13x + 3y - 4z = \cancel{0}\,4\end{gathered}}$$

3. **(10 points)** Let $\mathbf{F}(x,y,z) = (xy^2, z\cos(y), x+y+z)$. Calculate the curl and the divergence of $\mathbf{F}$.

$$\operatorname{curl}\vec{F} = (1-\cos y, -1, -2xy)$$

$$\operatorname{div}\vec{F} = y^2 - z\sin y + 1$$

4. **(10 points)** Let C be the curve $r(t) = (t, t^2, t^3)$ and let $\mathbf{F}(x,y,z) = (xy, x-y, z)$. Calculate the work done by $\mathbf{F}$ moving a particle along the curve from $(0,0,0)$ to $(2,4,8)$.

$t=0 \quad t=2$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^2 \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$= \int_0^2 (t^3, t-t^2, t^3)\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^2 t^3+2t^2-2t^3+3t^5 = \int_0^2 3t^5-t^3+2t^2\,dt$$

$$= \frac{1}{2}t^6 - \frac{1}{4}t^4 + \frac{2}{3}t^3\Big|_0^2$$

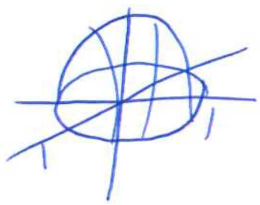
$$= 32-4+16/3$$

$$= 28+16/3 = 100/3$$

5. **(15 points)** Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

$$\mathbf{F}(x,y,z) = (2x^3+y^3)\mathbf{i} + (y^3+z^3)\mathbf{j} + 3y^2z\mathbf{k}$$

and S is the surface of the solid bounded by the parabloid $z = 1-x^2-y^2$ and the xy-plane.



$\operatorname{div}\vec{F} = 6x^2+3y^2+3y^2 = 6x^2+6y^2$

Want $\iiint_E 6x^2+6y^2\,dV$. Use cylindrical.

$0\le\theta\le 2\pi,\quad 0\le r\le 1\quad 0\le z\le 1-r^2$

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^2\cdot r\,dz\,dr\,d\theta$$

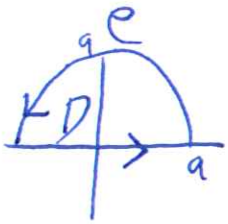
$$= \int_0^{2\pi}\int_0^1 6r^3 z\Big|_0^{1-r^2} dr\,d\theta = \int_0^{2\pi}\int_0^1 6r^3(1-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{3}{2}r^4 - r^6\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\,d\theta = \pi$$

6. **(15 points)** Let C be the boundary of the half disk $x^2+y^2 \le a^2$, $y \ge 0$, oriented counterclockwise. Use Green's Theorem to evaluate:

$$\oint_C (\sin x + 3y^2)dx + (2x - e^{-y^2})dy.$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - 6y \text{ so we want } \iint_D 2-6y\, dA.$$

Using polar: $0 \le \theta \le \pi$, $0 \le r \le a$

$$\int_0^{\pi}\int_0^a (2-6r\sin\theta) r\,dr\,d\theta = \int_0^{\pi}\int_0^a 2r - 6r^2\sin\theta\, dr\,d\theta$$

$$= \int_0^{\pi} r^2 - 2r^3\sin\theta \Big|_0^a d\theta$$

$$= \int_0^{\pi} a^2 - 2a^3\sin\theta\, d\theta = a^2\theta + 2a^3\cos\theta \Big|_0^{\pi}$$

$$= \cancel{2\pi a^2 + 2a^3 - (0-2a^3)}$$

$$= \cancel{2\pi a^2 + 4a^3}$$

$$= (\pi a^2 - 2a^3) - (0 + 2a^3)$$

$$= \boxed{\pi a^2 - 4a^3}$$

$$\boxed{\pi a^2 - 4a^3}$$

7. **(15 points)** Let S be the helicoid parameterized by $\mathbf{r}(u,v) = (u\cos v, u\sin v, v)$, $0 \le u \le 1$, $0 \le v \le \pi/2$.

a. Set up but do not evaluate an integral which gives the surface area of S.

b. Let $\mathbf{F}(x,y,z) = (z,y,x)$. Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

$r_u = (\cos v, \sin v, 0)$ $\quad r_v = (-u\sin v, u\cos v, 1)$ $\quad r_u \times r_v = (\sin v, -\cos v, u)$

$|r_u \times r_v| = \sqrt{1+u^2}$

a. Surface area $= \int_0^1 \int_0^{\pi/2} \sqrt{1+u^2}\, dv\, du$

b $\int_0^1 \int_0^{\pi/2} \vec{F}(\vec{r}(u,v)) \cdot (r_u \times r_v)\, dv\, du = \int_0^1 \int_0^{\pi/2} (v, u\sin v, u\cos v)\cdot(\sin v, -\cos v, u)\, dv\, du$

$= \int_0^1 \int_0^{\pi} v\sin v - u\sin v\cos v + u^2\cos v\, dv\, du$

$= \int_0^1 -v\cos v + \sin v - \frac{u}{2}\sin v + u^2 \sin v \Big|_0^{\pi/2} du$

$= \int_0^1 (0+1-\frac{u}{2}+u^2) - (0+0+0)\, du$

$= u - \frac{u^2}{4} + \frac{u^3}{3}\Big|_0^1 = 1 - 1/4 + 1/3$

$= \boxed{13/12}$