Math 241S- Midterm #2- November 10, 2015

NAME: SOLUTIONS

Flip over and put name on back of exam also

Problem #	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

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Instructions: You may not use any outside help, including notes, index cards, electronic devices or your neighbor!

1. **(20 points)** Short answer, very limited partial credit.

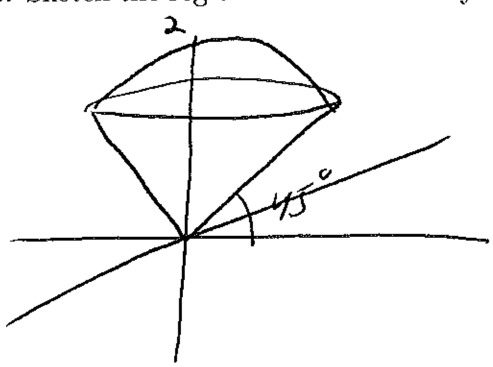
a. Suppose $f(x,y) = x^2y + 3y$. Find the directional derivative of $f(x,y)$ at the point $(2,3)$ in the direction of the vector $\vec{v} = (1,2)$.

$\nabla f = (2xy,\ x^2+3)$ $\quad \nabla f(2,3) = (12,7)$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = (1/\sqrt{5},\ 2/\sqrt{3})$

$\nabla f(2,3) \cdot \vec{u} = 26/\sqrt{5}$

b. Sketch the region in $\mathbb{R}^3$ describe by the spherical equations $0 \leq \phi \leq \pi/4$ and $0 \leq \rho \leq 2$.



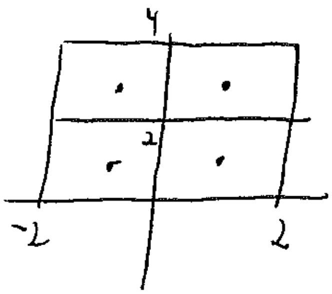
c. Let $f(x,y,z) = xyz$. Calculate the gradient ∇f.

$\nabla f = (yz,\ xz,\ xy)$

d. Let R be the rectangle: $\{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4\}$. Estimate

$$\iint_R x^2 y \, dA$$

using a Riemann sum with 4 rectangles (i.e. $m = n = 2$) and using the midpoint rule.



$\Delta A = 2 \times 2 = 4$

$$\left(f(-1,1) + f(1,1) + f(-1,3) + f(1,3)\right) \cdot 4$$

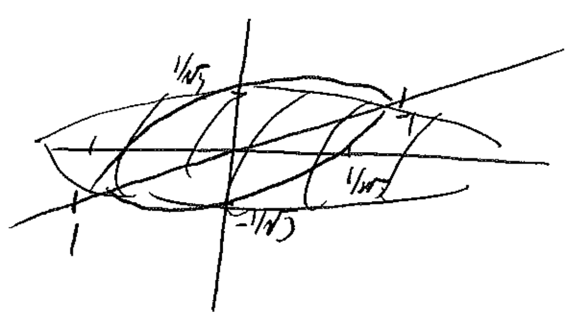
$$= (1+1+3+3)4 = 32$$

e. Describe and give a rought sketch of the region E in $\mathbb{R}^3$ that makes

$$\iiint_E 1 - x^2 - 2y^2 - 3z^2 dV$$

as large as possible.

$x^2 + 2y^2 + 3z^2 \le 1$



ellipsoid

2. **(10 points)** Let $z = (x-y)^5, x = s^2t, y = st^2$. Use the Chain Rule to find $\partial z/\partial s$. Express your answer in terms of s and t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= 5(x-y)^4 \cdot 2st - 5(x-y)^4(t^2)$$

$$= \boxed{5(s^2t-st^2)^4(2st-t^2)}$$

3. **(10 points)** Let $f(x, y) = xe^y$. Find the maximum and minimum value for $f(x, y)$ subject to the constraint $x^2 + y^2 = 2$.

Use Lagrange Mult

$\nabla f = (e^y, xe^y)$ $\quad \nabla g = (2x, 2y)$ $\quad$ so

$e^y = 2\lambda x$ $\qquad$ If $\lambda = 0$ then $e^y = 0$ ✱

If $x = 0$ then $y = 0$ ✱ so $x \neq 0$

$xe^y = 2\lambda y$ $\qquad e^y = 2\lambda x = 2\lambda \frac{y}{x}$

$x^2 + y^2 = 2$

so $x = \frac{y}{x} \Rightarrow y = x^2$

$y + y^2 = 2$

$y^2 + y - 2 = 0 \quad (y+2)(y-1) = 0 \Rightarrow y = -2, \; y = 1$

$y = -2$ gives no solutions to $x^2 + y^2 = 2$

$y = 1$ gives $x = \pm 1$

point	f	
$(1,1)$	e	$\leftarrow$ max value e
$(-1,1)$	$-e$	$\leftarrow$ min value $-e$

4. **(10 points)** Let
$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$
Find the critical points for $f(x, y)$ and use the second derivative test to classify each as a local max, min or saddle points.

$\nabla f = (3 - 3x^2, -4y + 4y^3)$ set $= (0,0)$

$x = \pm 1 \quad y = \pm 1$ or 0 6 points.

$f_{xx} = -6x \quad f_{yy} = 12y^2 - 4 \quad f_{xy} = 0$

$D = (-6x)(12y^2 - 4)$

point	D	f_{xx}	classify
(1,1)	-48		saddle
(1,-1)	-48		saddle
(1,0)	24	-6	local max
(-1,1)	48	6	local min
(-1,-1)	48	6	local min
(-1,0)	-24		saddle

5. **(10 points)** Find the equation of the tangent plane and normal line to the surface $xy^2z^3 = 8$ at the point $(2, 2, 1)$.

level curve $F=8$, $F=xy^2z^3$

∇F is $\perp$ to level curve

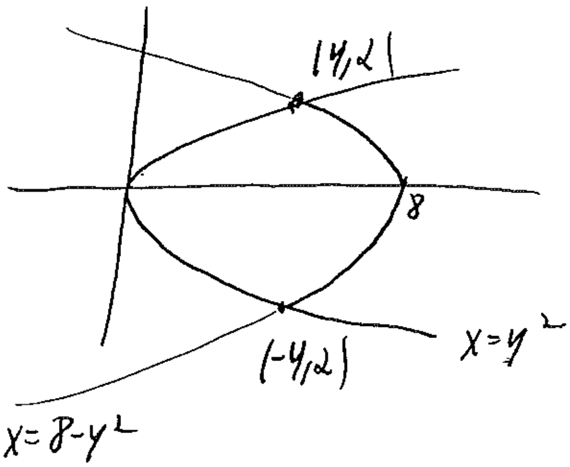
$$\nabla F = (y^2z^3, 2xyz^3, 3xy^2z^2)$$

$$\nabla F(2,2,1) = (4, 8, 24)$$

$$\boxed{4x+8y+24z = 48}$$

Normal line $\boxed{\vec{r}(t) = (2,2,1) + t(4,8,24)}$

6. **(10 points)** Let D be the region in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$. Find the area of D.



$y^2 = 8 - y^2 \Rightarrow y = \pm 2$

$$\text{Area} = \iint_D 1\,dA = \int_{-2}^{2}\int_{y^2}^{8-y^2} 1\,dx\,dy = \int_{-2}^{2} 8 - 2y^2\,dy$$

$$= 8y - \tfrac{2}{3}y^3 \Big|_{-2}^{2}$$

$$= \left(16 - \tfrac{16}{3}\right) - \left(-16 + \tfrac{16}{3}\right)$$

$$= 32 - \tfrac{32}{3} = 64/3$$

Final answer should be 32/3 and I should only have integrated from 0 to 2 since the problem says first quadrant!

7. **(10 points)** Let H be the solid hemisphere that lies above the xy-plane, has center $(0,0,0)$ and radius 1. Evaluate:

$$\iiint_H z\sqrt{x^2+y^2+z^2}\,dV.$$

Do in spherical

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/2$

$0 \leq \rho \leq 1$

$f(x,y,z) = \rho\cos\phi \cdot \rho$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cos\phi \cdot \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\sin\phi\cos\phi \Big|_{\rho=0}^{1} d\phi\, d\theta$$

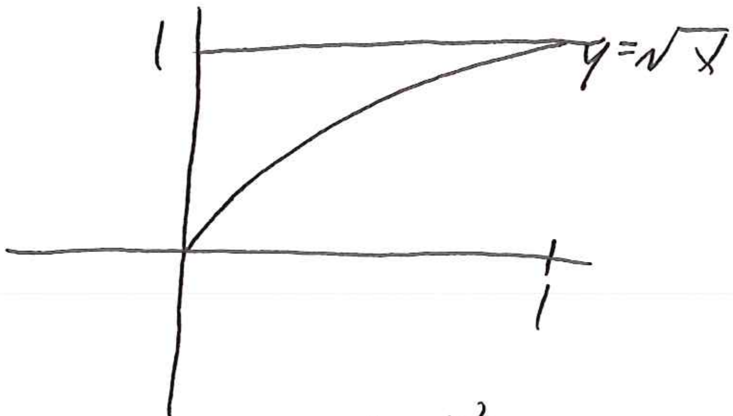
$$= \frac{1}{5}\int_0^{2\pi} \frac{\sin^2\phi}{2}\Big|_{\phi=0}^{\pi/2} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \tfrac{1}{2}\, d\theta = \pi/5$$

8. **(10 points)** Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\, dy dx$$

by changing the order of integration.



$0 \le y \le 1$

$0 \le x \le y^2$

$$\int_0^1 \int_0^{y^2} \sqrt{y^3+1}\, dx\, dy$$

$$= \frac{1}{3}\int_0^1 3 \cdot y^2 \sqrt{y^3+1}\, dy$$

$u = y^3+1$

$du = 3y^2\, dy$

$$= \frac{1}{3} \cdot u^{3/2} \cdot \frac{2}{3} =$$

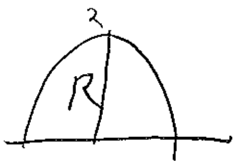
$$\frac{2}{9}(y^3+1)^{3/2} \Big|_0^1$$

$$= \frac{2}{9}\left(2^{3/2} - 1\right)$$

9. **(10 points)** Let R be the top half of the disc of radius 2 centered at the origin, i.e.:

$$\begin{aligned} 0 &\le x^2 + y^2 \le 4 \\ 0 &\le y \end{aligned}$$

Let $f(x, y) = y$. Calculate the average value of the function $f(x, y)$ on the region D.



$\text{Area } R = \frac{1}{2}\pi r^2 = 2\pi$

$\text{Avg value} = \frac{1}{\text{Area} R} \iint_R y \, dA$ do in polar!

$$\int_0^{\pi}\int_0^{2} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{\pi} \frac{r^3}{3}\sin\theta \Big|_0^2 \, d\theta = \int_0^{\pi} 8/3 \sin\theta \, d\theta$$

$$= -8/3 \cos\theta \Big|_0^{\pi}$$

$$= \frac{8}{3} - (-8/3) = 16/3$$

$$\text{Avg value} = \frac{16/3}{2\pi} = \frac{8}{3\pi}$$

NAME: