SOLUTIONS Math 241S- Midterm Exam #1 - October 1, 2015

Instructions: You may not use any outside help, including notes, index cards, electronic devices or your neighbor!

1. **(24 points)** Short answer, no partial credit. Let $\vec{u} = (2, 3, 4)$ and $\vec{v} = (-1, 1, 3)$.
   a. Calculate $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$.

$$\boxed{\begin{gathered} u \cdot v = -2 + 3 + 12 = 13 \\ u \times v = (15, -10, 5) \end{gathered}}$$

   b. Find the magnitude of $\vec{u} - 3\vec{v}$.

$$\vec{u} - 3\vec{v} = (5, 0, -5)$$

$$|\ | = \sqrt{50}$$

   c. Find a unit vector parallel to $\vec{u}$ ~~and pointing in the opposite direction.~~

$$\frac{-\vec{u}}{|u|} = \boxed{\left(\frac{-2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-4}{\sqrt{29}} \right)}$$

   d. Find the angle between $\vec{u}$ and $\vec{v}$,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{13}{\sqrt{29}\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{29}\sqrt{11}} \right)$$

e. Find the area of the triangle with vertices $P = (1,0,2), Q = (0,0,3), R = (1,2,3)$.

$\vec{PQ} = (-1, 0, 1)$

$\vec{PR} = (0, 2, 1)$

$\vec{PQ} \times \vec{PR} = (-2, 1, -2)$

$\frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{9} = 3/2$

f. State Clairaut's theorem.

Suppose $f(x,y)$ is defined on a disc D containing (a,b)

If f_{xy} & f_{yx} are continuous on D then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

g. Find the work done by a force $\mathbf{F} = 6\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ that moves an object from the point $(0, 8, 6)$ to the point $(4, 12, 20)$ along a straight line. The distance is measured in meters and the force in Newtons.

$D = (4, 4, 14)$

$W = \vec{F} \cdot \vec{D} = (6, -8, 7) \cdot (4, 4, 14)$

$= 24 - 32 + 98 = 90$ joules

h. Let $\vec{a} = (2, -1, 6)$, $\vec{b} = (1, 3, 4)$. Find the scalar projection of b onto a, i.e. $\text{comp}_{\vec{a}}\,\vec{b}$

$$\vec{b} \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{23}{\sqrt{4+1+36}} = \frac{23}{\sqrt{41}}$$

2. **(16 points)** Consider the surface given by the graph $z = x^2 + 2y^2$.

a. Let $\vec{r}_1(t) = (t, t+1, 3t^2 + 4t + 2)$ and $\vec{r}_2(t) = (2, t, 4 + 2t^2)$. Verify both curves lie on the surface.

b. Find the tangent vectors to each curve at the point $(2, 3, 22)$.

c. Find the equation of the plane containing the point $(2, 3, 22)$ and parallel to the two tangent vectors. This gives an alternate means for finding the tangent plane to the surface.

d. Sketch the surface neatly. Also on the same sketch include the curve $r_2(t)$ and its tangent vector at (2,3,22).

a. $3t^2+4t+2 = t^2+2\cdot(t+1)^2$ so $r_1(t)$ is on $z=x^2+2y^2$

$4+2t^2 = 2^2+2t^2$ so $r_2(t)$ " " "

b. $\vec{r}_1'(t) = (1, 1, 6t+4)$

$r_1'(2) = \boxed{(1,1,16)}$

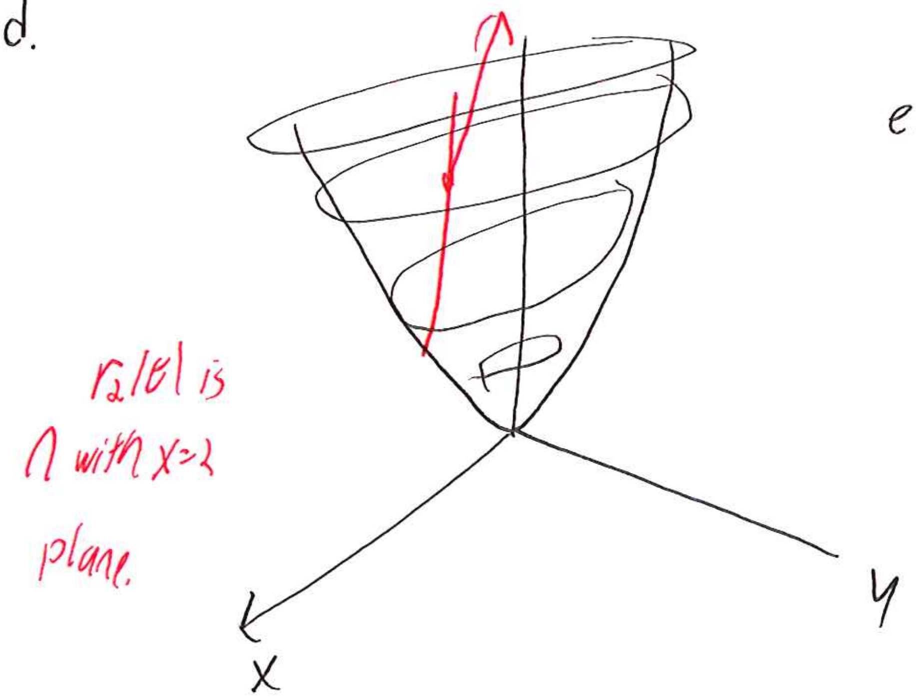
$r_2'(t) = (0, 1, 4t)$

$r_2'(3) = \boxed{(0,1,12)}$

c. Use $\vec{n} = (1,1,16) \times (0,1,12) = (-4,-12,1)$

$$\boxed{\begin{gathered}(-4,-12,1)\cdot(x-2, y-3, z-22) = 0 \\ \text{OR} \\ -4x-12y+z = -22\end{gathered}}$$

d.

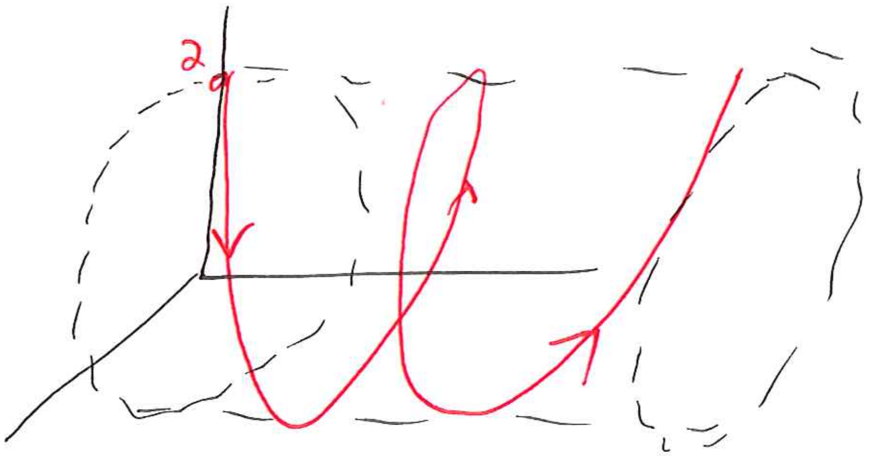


elliptic paraboloid

3. **(10 points)** Let $\vec{r}(t) = (2\sin(2t), t, 2\cos(2t))$ represent the position of a particle at time t.

a. Sketch $\vec{r}(t)$ for $0 \leq t \leq 2\pi$, with an arrow in the direction of increasing t.

Notice $\vec{r}(t)$ lies on cylinder $x^2 + z^2 = 4$



Curve starts at $(0,0,2)$ & winds around twice

b. Find the tangent vector to the space curve $\vec{r}(t)$ at $t = \pi/2$.

$\vec{r}'(t) = (4\cos 2t, 1, -4\sin 2t)$

$\vec{r}'(\pi/2) = (-4, 1, 0)$

c. Find the length of the curve sketched in part a.

speed $= |r'(t)| = \sqrt{16\cos^2 2t + 1 + 16\sin^2 2t}$

$= \sqrt{17}$

$A.L. = \int_0^{2\pi} \sqrt{17}\,dt = 2\pi\sqrt{17}$

4. **(10 points)** a. Find an equation of the plane through the point $(5, 0, 3)$ and perpendicular to the line $x = 6t$, $y = 4t$, $z = 8 + 4t$.

b. Find an equation for the plane consisting of all points that are equidistant from the points $(5, 0, 2)$ and $(7, 4, 0)$.

a. $\vec{n} = \langle 6, 4, 4 \rangle$

$$\boxed{\begin{gathered} \langle 6,4,4\rangle \cdot \langle x-5, y, z-3\rangle = 0 \\ \text{or} \\ 6x + 4y + 4z = 42 \end{gathered}}$$

b. midpoint $= \langle 6, 2, 1\rangle$ is on plane.

normal is $\langle 7,4,0\rangle - \langle 5,0,2\rangle = \langle 2, 4, -2\rangle$

$$\boxed{\begin{gathered} \langle 2,4,-2\rangle \cdot \langle x-6, y-2, z-1\rangle = 0 \\ \text{or} \\ 2x + 4y - 2z = 18 \end{gathered}}$$

5. **(10 points)** At what point do the curves $\vec{r}_1(t) = (t, 2-t, 15+t^2)$ and $\vec{r}_2(s) = (5-s, s-3, s^2)$ intersect? Find their angle of intersection.

$t = 5-s$

$2-t = s-3$

$15+t^2 = s^2$

Sub 1 into 3 $\quad 15+(5-s)^2 = s^2$

$s^2 - 10s + 40 = s^2$

$s = 4 \quad t = 1$

$r_1(1) = r_2(4) = \boxed{(1,1,16)}$

$r_1' = (1,-1,2t) \qquad r_1'(1) = (1,-1,2)$

$r_2' = (-1,1,2s) \qquad r_2'(4) = (-1,1,8)$

$$\cos\theta = \frac{(1,-1,2)\cdot(-1,1,8)}{\sqrt{66}\sqrt{10}} = \frac{14}{\sqrt{66}\sqrt{10}}$$

$$\theta = \cos^{-1}\frac{14}{\sqrt{66}\sqrt{10}}$$

6. **(5 points)** Consider a function $f(x, y)$. Define, using limits the partial derivative $f_x(x, y)$.

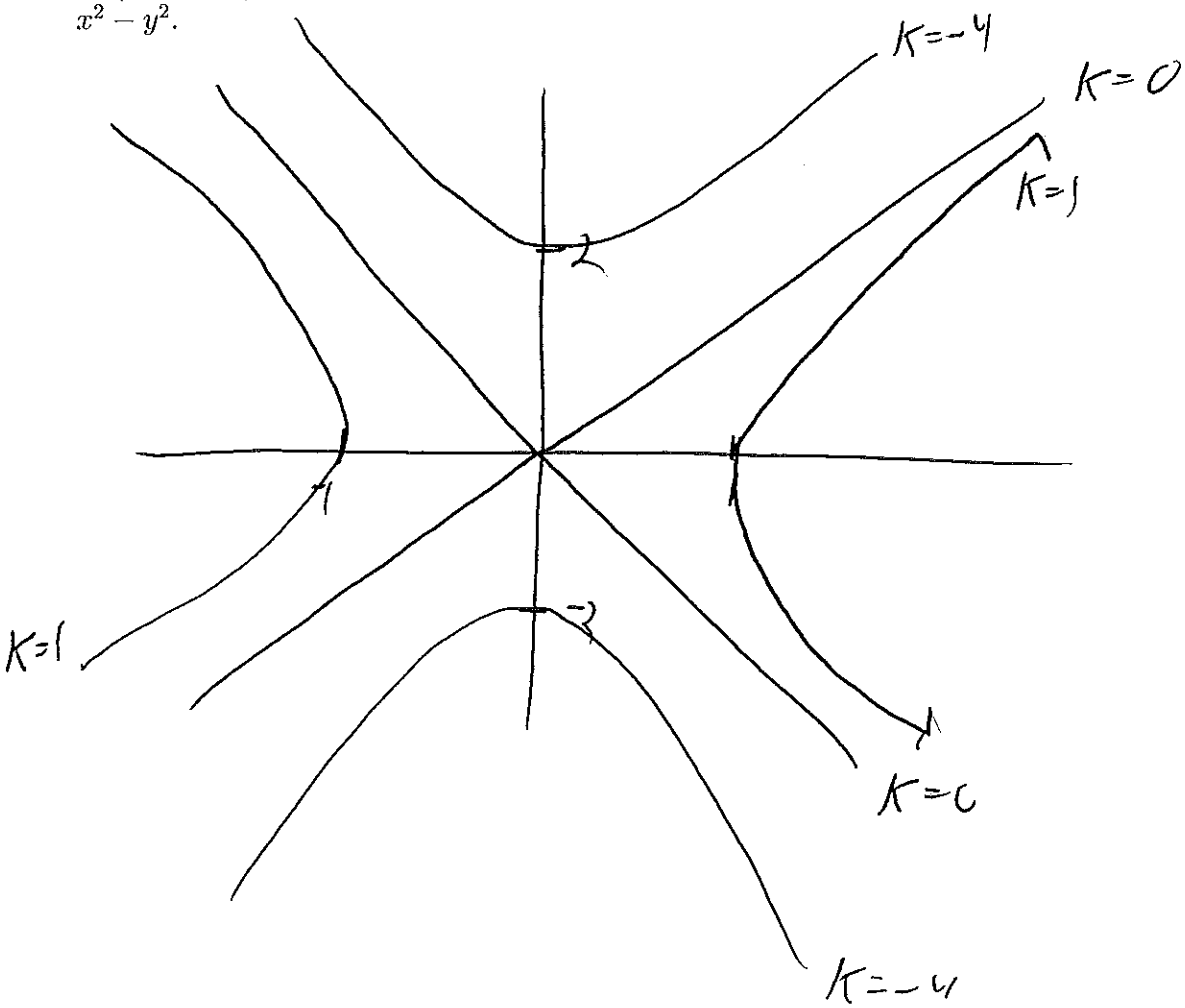
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

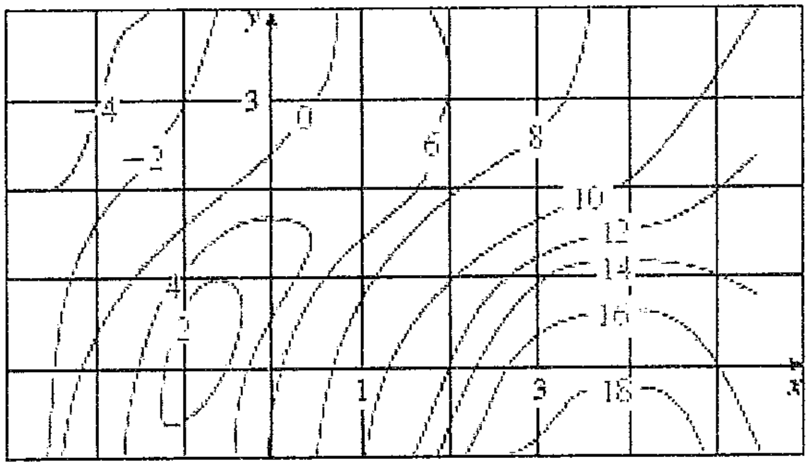
7. **(5 points)** Given $x^2z + y^2 + z^2 = 3$. Find the partial derivative $\frac{\partial z}{\partial x}$ using implicit differentiation.

$$2xz + x^2 \frac{dz}{dx} + 0 + 2z \frac{dz}{dx} = 0$$

$$\boxed{\frac{dz}{dx} = \frac{-2xz}{x^2 + 2z}}$$

8. (**5 points**) Sketch the level curves $f(x, y) = k$ for values $k = 0, 1$ and -4 where $f(x, y) = x^2 - y^2$.





9. (10 points)
Above is a countour plot for a function $f(x, y)$.
a. Estimate the partial derivatives $f_x(2,1)$ and $f_y(2,1)$. Show your work.

b. Is $f_{yy}(3,2)$ postive or negative? Explain

a. $f(2,1) = 10$ $f(1.5,1) \approx 8.9$ $f(2.5,1) \approx 11.5$

slope from $(1.5,1)$ to $(2,1)$ is $\frac{1.1}{.5}$

slope from $(2,1)$ to $(2.5,1)$ is $1.5/.5$

avg is
$2.6 \approx f_x(2,1)$

$f(2,1.5) \approx 9$ $f(2,1) = 10$ $f(2,.5) = 11$

slopes both -2

$-2 \approx f_y(2,1)$

b. $f_y(3,2)$ is negative because as y increases the elevation decreases

However the decrease is getting less so

$f_{yy}(3,2)$ is positive

NAME:

10. **(5 points)** Let $f(x,y) = \frac{xy^2}{x^2+y^4}$. Prove that:

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

On $x=0$, $f(x,y) = 0/y^4 = 0$

so if a limit exists, it must be zero, as every ball of radius δ around $(0,0)$ has points w/ $f=0$.

On $x=y^2$, $f(x,y) = 1/2$.

Every ball around $(0,0)$, no matter how small, has points where $f=0$ and where $f=1/2$.

$\therefore$ The limit DNE