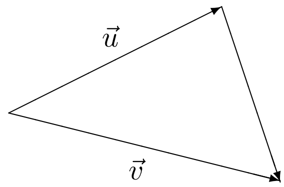
Math 2950- Final Exam - December 15, 2004

1. **(10 points)**
a. Find the angle between $(-1, 1, 0)$ and $(0, 1, -1)$.

b. Find the component (a.k.a. scalar projection) of $(2, 3, -4)$ in the direction of $(1, 1, \sqrt{2})$.

2. **(15 points)**
Consider the picture below:



a. Label the third vector shown in the picture.

b. Use vectors to show that the line segment connecting the midpoints of two sides of a triangle is parallel to the third side and has length half the length of the third side. (Hint: Draw the triangle with one vertex at the origin and consider the diagram above.)

3. **(10 points)** Find the equation of the plane (in any form you like) passing through $(2, -2, 5)$ and perpendicular to the line through $(1, 1, 1)$ and $(-2, 3, 1)$.

4. (**15 points**) A particle moves on the path $\vec{x}(t) = (2\cos(t), \sin(t), t)$. Find the velocity, acceleration and speed. Then neatly sketch and label the path between $t = 0$ and $t = 6\pi$.

5. **(10 points)** Let $\mathbf{F}(x, y, z) = (xyz, x\sin(y), yz)$. Calculate div$\mathbf{F}$ and curl$\mathbf{F}$.

6. **(15 points)**
a. $w = x^2 + y^2, x = s\cos(t), y = s\sin(t)$. Find $\frac{\partial w}{\partial s}$ at the point where $t = \pi/3$ and $s = 2$.

b. $z = x^2\cos(xy)$. Find the total differential dz.

7. **(15 points)** Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$ where $\mathbf{F}(x, y) = (x^2y, x^2 - y)$ and C is the curve $\mathbf{r}(t) = (t, 1-t)$, $0 \leq t \leq 1$.

8. **(15 points)** Consider the function $f(x, y) = x + y - 2\sqrt{xy}$ defined on the first quadrant $x \geq 0, y \geq 0$. Show the global minimum of $f(x, y)$ is 0,

i.e. $f(x, y) \geq 0$. (Hint: There will be an entire ray of critical points)

Use the result above to prove the arithmetic/geometric mean inequality:

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

9. **(15 points)** Find the absolute maximum and minimum values of $f(x, y) = 3 + xy - x - 2y$ on the closed triangular region with vertices $(0, 0), (5, 0), (0, 5)$.

10. **(15 points)** Calculate $\iiint_E xz\, dV$ where E is the region between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$, above the xy-plane (i.e. $z \geq 0$) and in front of the yz plane (i.e. $x \geq 0$).

11. **(20 points)**
a. Show that the vector field

$$\mathbf{F}(x, y, z) = (y + z, x + 2y, -\sin(z) + x)$$

is conservative by checking that curl$\mathbf{F} = 0$.

b. Find a potential function $f(x, y, z)$.

c. Let C be the curve parameterized

$$\mathbf{r}(t) = (te^t, \frac{\sqrt{t+1}}{5+t}, \pi t^2 \sqrt{t+1})$$

for $0 \leq t \leq 3$. Calculate the work done by the force $\mathbf{F}$ on a moving particle traversing C.

12. **(15 points)** Use Green's Theorem to evaluate:

$$\oint_C (x^2 - y^3)dx + (x^3 + y^2)dy$$

where C is the unit circle $x^2 + y^2 = 1$ traversed *clockwise.*

13. **(15 points)**
a. Find the surface area of the *helicoid*

$$\mathbf{r}(u, v) = (u\cos(v), u\sin(v), v), 0 \leq u \leq 1, 0 \leq v \leq \pi.$$

b. Find the tangent plane to the helicoid at the point where $u = 1/2$, $v = \pi/2$.

14. (**15 points**) Find the area enclosed by the ellipse $4x^2 + y^2 = 1$.