Name: SOLUTIONS

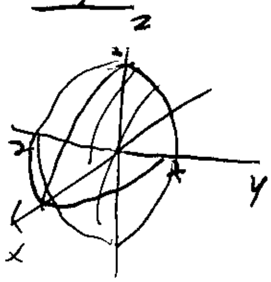
Math 241- Midterm Exam #3 - November 13, 2008

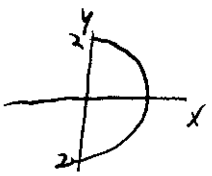
Instructions: You are allowed a single 3" by 5" index card but no other notes, books, or calculators.

1. **(15 points)** Use spherical coordinates to evaluate:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

Region "Front" half of sphere radius 2.





$$0 \le \rho \le 2$$

$$-\pi/2 \le \theta \le \pi/2$$

$$0 \le \phi \le \pi$$

$$\int_0^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \rho \cdot \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \rho^3 d\rho \int_{-\pi/2}^{\pi/2} 1\, d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{\rho^4}{4}\Big|_0^2 \cdot \theta\Big|_{-\pi/2}^{\pi/2} \; (-\cos\phi)\Big|_0^{\pi}$$

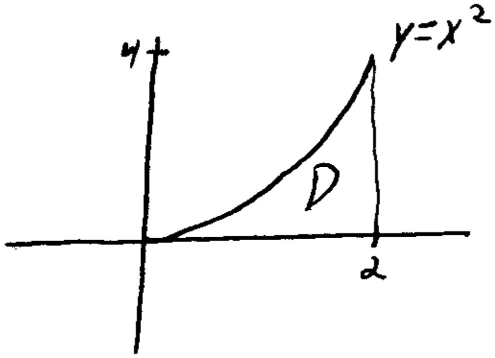
$$= 8 \cdot \pi \cdot (1 - -1) =$$

$\cancel{8\pi}$

16π

2. **(10 points)** Draw a sketch of the region D over which the iterated integral is being evaluated, then express it as an iterated integral in which the order of integration is reversed:

$$\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx.$$



$0 \le y \le 4$

$\sqrt{y} \le x \le 2$

$$\boxed{\int_0^4 \int_{\sqrt{y}}^2 f(x,y)\,dx\,dy}$$

3. **(15 points)** Find the work done by a force $\vec{F}(x,y) = (2-y, x)$ in moving a particle along one arch of the cycloid given by $\vec{r}(t) = (t - \sin t, 1 - \cos t)$ for $0 \le t \le 2\pi$.

$$\vec{r}'(t) = (1 - \cos t, \sin t)$$

$$\vec{F}(\vec{r}(t)) = (1 + \cos t, t - \sin t)$$

$$\vec{F}(r(t)) \cdot \vec{r}'(t) = (1 - \cos^2 t + t\sin t - \sin^2 t)$$

$$= 1 - (\sin^2 t + \cos^2 t) + t \sin t$$

$$= t \sin t$$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} t \sin t \, dt$$

$$= -t\cos t + \sin t \Big|_0^{2\pi}$$

$$= -2\pi + 0 - (\overset{0}{\cancel{2\pi}})$$

$$= -2\pi$$

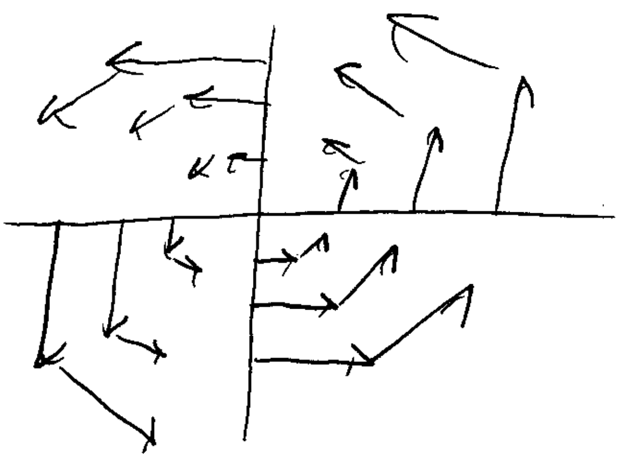
4. **(5 points)** A vector field $\vec{F}$ is *conservative* if ...

there is a function f with

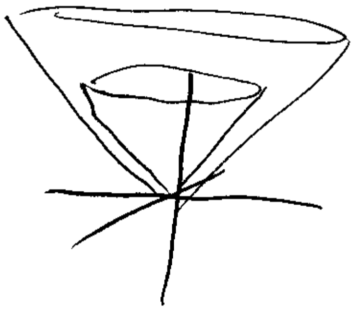
$$\nabla f = \vec{F}$$

5. **(10 points)** Neatly sketch the vector field below, be sure to sketch enough vectors so the behavior of the vector field is clear.

$$\vec{F}(x, y) = (-y, x)$$



6. **(10 points)** Find the volume of the part of the ball $\rho \leq 5$ that lies between the cones $\phi = \pi/4$ and $\phi = \pi/3$.



$$V = \int_0^5 \int_{\pi/4}^{\pi/3} \int_0^{2\pi} 1 \cdot \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

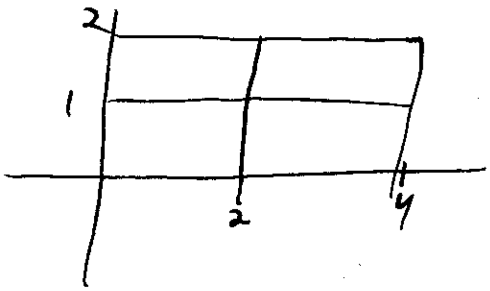
$$= \int_0^5 \rho^2 d\rho \int_{\pi/4}^{\pi/3} \sin\phi \, d\phi \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{\rho^3}{3}\Big|_0^5 \left(-\cos\phi\right)\Big|_{\pi/4}^{\pi/3} \theta\Big|_0^{2\pi}$$

$$= \frac{125}{3} \cdot \left(-\cos\tfrac{\pi}{3} + \cos\tfrac{\pi}{4}\right)\left(2\pi\right)$$

$$= \boxed{\frac{250\pi}{3}\left(-\tfrac{1}{2} + \tfrac{\sqrt{2}}{2}\right)}$$

7. **(10 points)** Let R be the rectangle $[0,4] \times [0,2]$. Estimate $\int\int_R x^2 y$ using a Riemann sum with $m = n = 2$ and the upper right corner of each rectangle as your sample point.

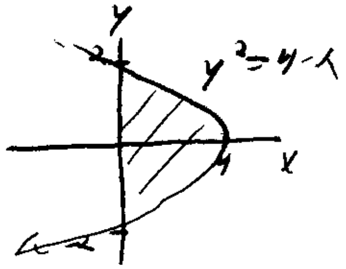


$\Delta x \Delta y = 2$

$\iint \approx 2 \cdot [f(2,1) + f(4,1) + f(2,2) + f(4,2)]$

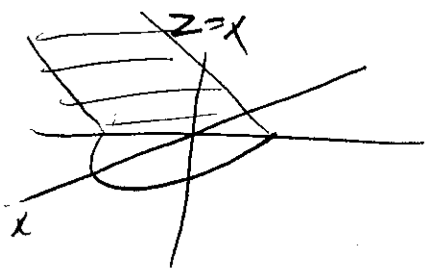
$= 2(4 + 16 + 8 + 32) = 120$

8. (**15 points**) Find the volume of the region $E \subset \mathbb{R}^3$ bounded below by the xy-plane, above by the plane $z = x$, and by the parabolic cylinder $y^2 = 4 - x$.



$0 \le x \le 4 - y^2$

$-2 \le y \le 2$



Volume =

$$\int_{-2}^{2}\int_{0}^{4-y^2}\int_{0}^{x} 1\,dz\,dx\,dy$$

$$= \int_{-2}^{2}\int_{0}^{4-y^2} x\,dx\,dy$$

$$= \int_{-2}^{2} \frac{x^2}{2}\Big|_{0}^{4-y^2} dy$$

$$= \int_{-2}^{2} \frac{(4-y^2)^2}{2}\,dy$$

$$= \int_{-2}^{2} \frac{1}{2}y^4 - 8y^2 + 16\,dy$$

$$= \frac{1}{10}y^5 - \frac{8}{3}y^3 + 16y\Big|_{-2}^{2}$$

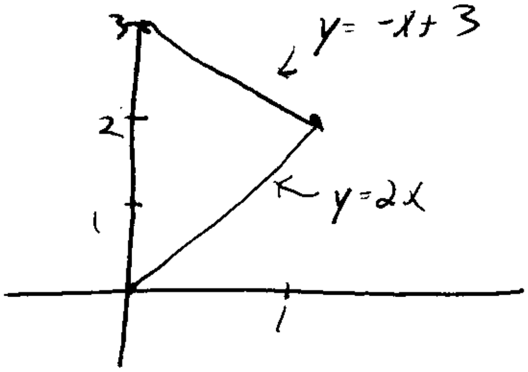
$$= \left(\frac{32}{10} - \frac{64}{3} + 32\right) - \left(-\frac{32}{10} + \frac{64}{3} - 32\right)$$

$$= 2\cdot\left(32 + 32/10 - 64/3\right)$$

9. (**10 points**) Evaluate the double integral:

$$\iint_D xy\,dA$$

where D is the triangular region with vertices $(0,0)$, $(1,2)$ and $(0,3)$.



$0 \le x \le 1$

$2x \le y \le 3-x$

$$\int_0^1 \int_{2x}^{3-x} xy\,dy\,dx$$

$$= \int_0^1 \frac{xy^2}{2}\Big|_{y=2x}^{y=3-x} dx$$

$$= \int_0^1 \frac{x}{2}\left((3-x)^2-(2x)^2\right)$$

$$= \int_0^1 \frac{x}{2}\left(x^2-6x+9-4x^2\right) dx$$

$$= \int_0^1 \frac{x^3}{2} - 3x^2 + 9/2 - 2x^3\,dx$$

$$= \frac{x^4}{8} - x^3 + 9/2\,x - \tfrac{1}{2}x^4 \Big|_0^1$$

$$= 1/8 - 1 + 9/2 - 1/2$$

$$= 1/8 + 3 = 25/8$$