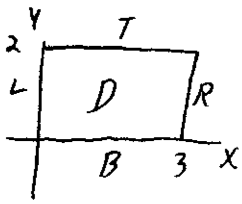
Name: SOLUTIONS

Math 241- Midterm Exam #2 - October 21, 2008

1. **(10 points)** Find the absolute maximum and minimum values of

$$f(x, y) = x^4 + y^4 - 4xy + 9$$

on the set $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.



$f_x = 4x^3 - 4y$ $\quad f_y = 4y^3 - 4x$, set both $= 0$ gives $y = x^3$, $x = y^3$ so $x = x^9$ so $x = 0, 1$ or -1. Thus

$\boxed{\text{Critical points } (0,0), (1,1), (-1,-1)}$.

On L, $x = 0$ so $f = y^4 + 9$ $\quad f' = 4y^3$, no C.P. test endpoints $(0,0)$, $(0,2)$

On B, $y = 0$ so $f = x^4 + 9$ $\quad f' = 4x^3$, no CP test endpoints $(0,0)$, $(3,0)$

On R, $x = 3$ so $f = 81 + y^4 - 12y + 9$, $\quad f' = 4y^3 - 12$

$4y^3 - 12 = 0$
$y^3 = 3$
$y = \sqrt[3]{3}$ $\quad$ test $(3, \sqrt[3]{3})$, $(3,0)$, $(3,2)$

On T, $y = 2$ so $f = y^4 + 16 - 8x + 9$, $\quad f' = 4x^3 - 8$ $\quad$ test $(\sqrt[3]{2}, 2)$, $(0,2)$, $(3,2)$

$4x^3 - 8 = 0$
$x = \sqrt[3]{2}$

point	f value
$(0,0)$	9
$(1,1)$	7
$(-1,-1)$	7
$(0,2)$	25
$(3,2)$	82
$(3,0)$	90
$(3,\sqrt[3]{3})$	$81 + 3\sqrt[3]{3} - 12\sqrt[3]{3} + 9 = 90 - 9\sqrt[3]{3}$
$(\sqrt[3]{2},2)$	$2\sqrt[3]{2} + 16 - 8\sqrt[3]{2} + 9 = 25 - 6\sqrt[3]{2}$

both > 7 and < 90

$$\boxed{\begin{array}{l}\text{Max value} = 90 \\ \text{Min value} = 7\end{array}}$$

2. **(10 points)** Let $f(x,y) = x^3 - 3x + y^4 - 2y^2$. Find all the critical points of $f(x,y)$ and classify each as a local maximum, local minimum or saddle point.

$f_x = 3x^2 - 3$, $f_y = 4y^3 - 4y$, $f_{xx} = 6x$, $f_{yy} = 12y^2 - 4$, $f_{xy} = f_{yx} = 0$

1. Find C.P. Need $f_x = 0$, $f_y = 0$ gives $x = \pm 1$, $y = 0, 1, -1$

so $\boxed{(1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1) \text{ are all C.P.}}$

2. $D = 6x \cdot (12y^2 - 4) - 0 = 6x(12y^2 - 4)$

C.P.	D	f_{xx}	Classify
(1,0)	−24	N/A	saddle
(1,1)	48	6	local min
(1,−1)	48	6	local min
(−1,0)	24	−6	local max
(−1,1)	−48	N/A	saddle
(−1,−1)	−48	N/A	saddle

3. **(10 points)** Find the maximum rate of change of $f(x,y) = xy^2 + \sqrt{x}$ at the point $(1,3)$. In what direction does it occur?

$\nabla f = \left(y^2 + \frac{1}{2\sqrt{x}},\ 2xy \right)$

$\nabla f(1,3) = \left(9 + \frac{1}{2},\ 6\right) = \boxed{\left(\frac{19}{2}, 6\right) \leftarrow \text{direction}}$

$|\nabla f(1,3)| = \boxed{\sqrt{\left(\frac{19}{2}\right)^2 + 36} \leftarrow \text{max rate}}$

4. **(10 points)** Consider the surface given by

$$xy + xz + y^2 z + 7x = 19.$$

Find the equation of the tangent plane and the normal line to this surface at the point $(1,2,2)$.

This is level surface $F(x,y,z) = 19$

$F_x = y + z + 7 \qquad F_y = x + 2yz \qquad F_z = x + y^2$

Now plug in $(1,2,2)$

$F_x = 11 \quad F_y = 9 \quad F_z = 5 \leftarrow$ normal vector

$$\boxed{\begin{array}{ll} \text{Tangent plane:} & 11(x-1) + 9(y-2) + 5(z-2) = 0 \\ \text{Normal line:} & (1,2,2) + t(11,9,5) \end{array}}$$

5. **(10 points)** Let

$$z^2 + \cos(x) + \frac{y}{z} = 5.$$

Find $\frac{\partial z}{\partial y}$.

$$F_y = 1/z \qquad F_z = 2z - \frac{y}{z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-1/z}{2z - y/z^2} = \boxed{\frac{-z}{2z^3 - y}}$$

6. **(10 points)** Let $f(x, y) = x\sin(xy) + y^3$. Find the directional derivative of $f(x, y)$ at the point $(\pi/2, 1)$ in the direction $(1, 4)$.

$$\vec{u} = \frac{(1,4)}{|(1,4)|} = (1/\sqrt{17}, 4/\sqrt{17})$$

$$\nabla f = (\sin(xy) + xy\cos(xy),\ x^2\cos(xy) + 3y^2)$$

$$\nabla f(\pi/2, 1) = (\sin(\pi/2) + \pi/2\cos(\pi/2),\ \pi^2/4\cos(\pi/2) + 3)$$

$$= (1 + 0, 0 + 3) = (1, 3)$$

$$D_{\vec{u}} = (1,3) \cdot (1/\sqrt{17}, 4/\sqrt{17}) = 13/\sqrt{17}$$

7. **(10 points)** Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}.$$

If the limit does not exist, write DNE but be sure to justify your answer.

On the line $y=x$, $f(x,y)= \frac{x^2+x^2}{x^2} = \frac{2x^2}{x^2} = 2$

On the line $y=-x$ $f(x,y)= \frac{2x^2}{-x^2} = -2$,

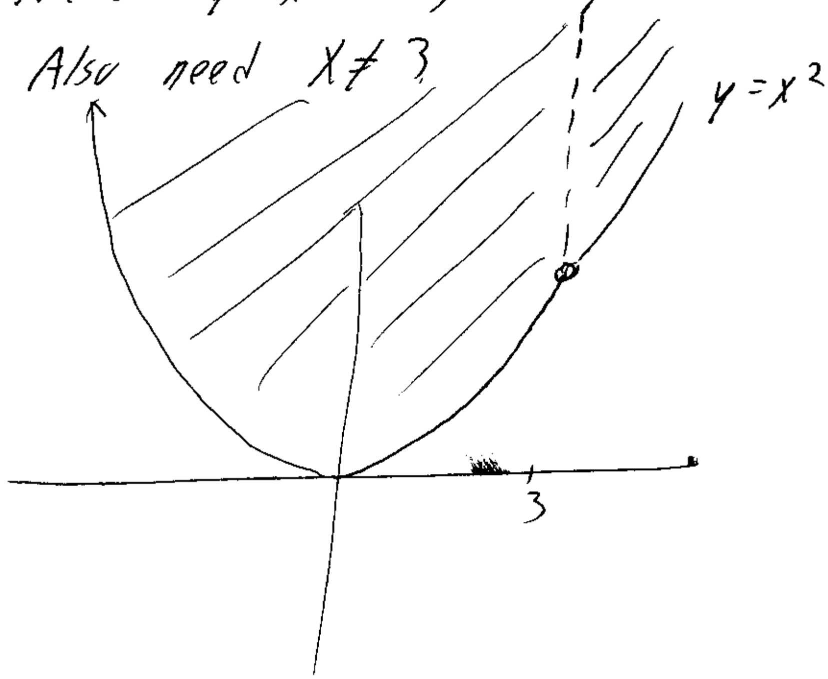
If the limit was $=L$ then the function would approach L along <u>any</u> curve approaching $(0,0)$.

Thus limit DNE.

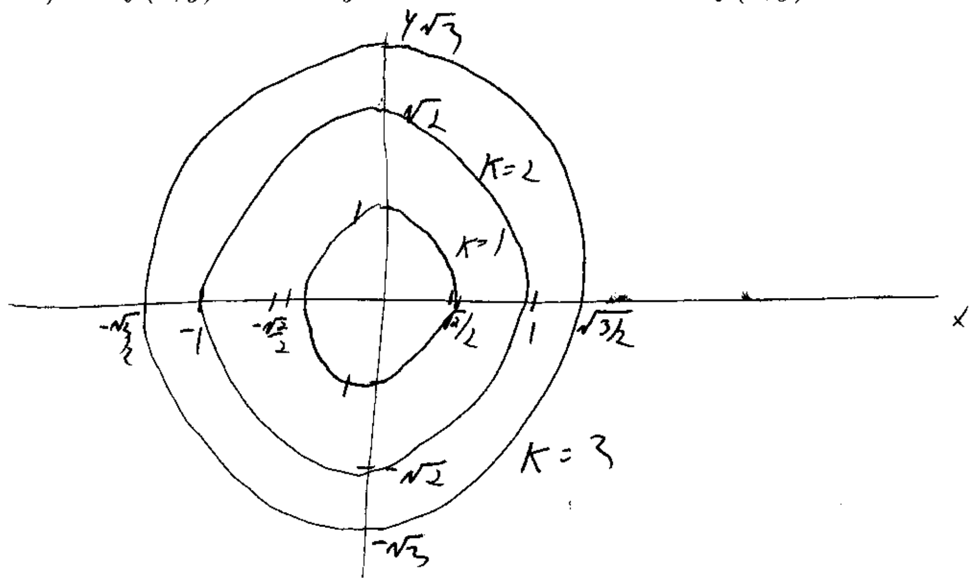
8. (**10 points**) Let $f(x,y) = \frac{\sqrt{y-x^2}}{x-3}$. Neatly sketch the domain of $f(x,y)$.

Need $y-x^2 \geq 0$, i.e. $y \geq x^2$

Also need $x \neq 3$



9. (**10 points**) Let $f(x,y) = 2x^2 + y^2$. Sketch the level curves $f(x,y) = k$ for $k = 1, 2, 3$.



10. (**10 points**) Suppose $F(x,y) = x^2y + y^2$, $x = st + v^2 + uv$, $y = s - u^2v$.

a. Find $\frac{\partial F}{\partial u}$.

b. Find $\frac{\partial F}{\partial s}$ when $s = 1, t = 2, u = 3, v = 4$.

a. $\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} = \boxed{(2xy)(v) + (x^2+2y)(-2uv)}$

b. $\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} = (2xy)(t) + (x^2+2y)(1)$

$x = 18 \quad y = -35$

$\frac{\partial F}{\partial s}$ at this point is

$2\cdot(18)(-35) + (18^2 - 70) = -1006$