Name: SOLUTIONS

Math 241- Midterm Exam #1 - September 18, 2008

1. **(14 points (2 pts each, no partial credit))** Let $\vec{a} = (-4, 1, 2)$ and $\vec{b} = (1, 2, 3)$.

a. Find $\vec{a} \times \vec{b}$.

$(-1, 14, -9)$

b. Find $\vec{a} \cdot \vec{b}$.

$-4 + 2 + 6 = 4$

c. Determine the magnitudes $|\vec{a}|$ and $|\vec{b}|$.

$|\vec{a}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$

$|\vec{b}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

d. Let Θ be the angle between $\vec{a}$ and $\vec{b}$. Find $\cos\Theta$.

$$\cos\Theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \boxed{\frac{4}{\sqrt{21}\sqrt{14}}}$$

e. Find $3\vec{a} - 2\vec{b}$.

$(-14, -1, 0)$

f. Find the vector projection $\text{prog}_{\vec{a}}\vec{b}$ of $\vec{b}$ onto $\vec{a}$.

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{4}{21}(-4,1,2) = \boxed{\left(-\tfrac{16}{21}, \tfrac{4}{21}, \tfrac{8}{21}\right)}$$

g. Find the area of the triangle with corners $(0,0,0)$, $(-4,1,2)$, $(1,2,3)$.

$$\text{area} = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\sqrt{1^2+14^2+9^2} = \boxed{\frac{\sqrt{278}}{2}}$$

2. **(5 points)** Given points $A = (1,1,1)$, $B = (2,3,0)$, $C = (-1,1,4)$ and $D = (0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC and AD.

$$\vec{AB} = (1,2,-1) \quad \vec{AC} = (-2,0,3) \quad \vec{AD} = (-1,2,1)$$

$$\text{volume} = |\vec{AB}\cdot(\vec{AC}\times\vec{AD})| = |(1,2,-1)\cdot(-6,-1,-4)|$$

$$= |-6-2+4| = |-4|$$

$$= 4$$

A B

3. **(10 points)** Find the equation of the plane containing the points $(2, 1, 1)$, $(3, 0, 2)$ and C $(-1, 1, 1)$. Then find the parametric equation of the line passing through $(3, 0, 2)$ and perpendicular to the plane.

1. $\vec{AB} = (1, -1, 1)$ $\vec{AC} = (-3, 0, 0)$

normal vector is $\vec{AB} \times \vec{AC} = (0, -3, -3)$

$(x-2, y-1, z-1) \cdot (0, -3, -3) = 0$

$-3y + 3 - 3z + 3 = 0$

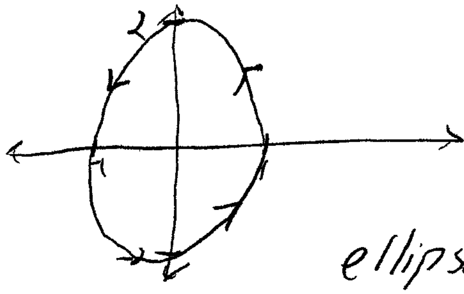
$\boxed{y + z = 2}$

2. Direction vector is $\vec{n}$ which is $(0, 1, 1)$

$\boxed{(x, y, z) = (3, 0, 2) + t(0, 1, 1)}$

4. **(5 points)** Sketch the curve $\vec{r}(t) = (\cos(t), 2\sin(t))$ for $0 \leq t \leq 2\pi$ in the xy-plane. Be sure to label intercepts and indicate with an arrow the direction of increasing t.

Notice curve is $- x^2 + \frac{y^2}{4} = 1$



ellipse

5. **(10 points)** The position function of a particle is given by $\vec{r}(t) = (t^2, 5t, t^2 - 16t)$ At what time is its speed a minimum? Hint: Minimizing the square of the speed is easier and clearly gives the same answer.

$\vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle$

speed $= |r'(t)| = \sqrt{(2t)^2 + 5^2 + (2t-16)^2}$

Let $f(t) = |r'(t)|^2 = 4t^2 + 25 + (2t-16)^2$

We minimize $f(t)$ by setting $f'(t) = 0$.

$f'(t) = 8t + 2(2t-16)\cdot 2$

$= 8t + 8t - 64 = 16t - 64$

$16t - 64 = 0$ $\quad t = 4$

Notice that $f''(t) = 16 > 0$ so this is a min by 2^{nd} derivative tes

6. **(10 points)** A particle has acceleration $\vec{a}(t) = (1, -1, t)$, initial velocity $\vec{v}(0) = (1, 1, -1)$ and initial position $\vec{r}(0) = (1, -1, 1)$. Find the equation for its position $\vec{r}(t)$.

$$\vec{v}(t) = \int \vec{a}(t) = (t, -t, t^2/2) + \vec{v}_0$$

$$\vec{v}(t) = (t+1, -t+1, t^2/2 - 1)$$

$$\vec{r}(t) = \int \vec{v}(t) = \left(\frac{t^2}{2} + t, -t^2/2 + t, t^3/6 - t\right) + \vec{r}_0$$

$$\boxed{\vec{r}(t) = \left(\frac{t^2}{2} + t + 1, -\frac{t^2}{2} + t - 1, \frac{t^3}{6} - t + 1\right)}$$

7. **(15 points)** Consider the space curve

$$\vec{r}(t) = (\cos(t), \sin(t), t^2).$$

a. Find the unit tangent vector and unit normal vector to the curve at the point $(0, 1, \frac{\pi^2}{4})$.
b. Determine the curvature $\kappa(t)$.
c. Set up but do not evaluate an integral which gives the length of the curve for $1 \le t \le 3$.

$\vec{r}'(t) = (-\sin t, \cos t, 2t)$ $\vec{r}''(t) = (-\cos t, -\sin t, 2)$ $|r'(t)| = \sqrt{1+4t^2}$

a. This point is when $t = \pi/2$. $T(t) = \dfrac{(-\sin t, \cos t, 2t)}{\sqrt{1+4t^2}}$

$$T(\pi/2) = \frac{\vec{r}'(\pi/2)}{|r'(\pi/2)|} = \boxed{\frac{(-1, 0, \pi)}{\sqrt{1+\pi^2}}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} \quad T'(t) = -\tfrac{1}{2}(1+4t^2)^{-3/2} \cdot 8t\,(-\sin t, \cos t, 2t) + \frac{1}{\sqrt{1+4t^2}}(-\cos t, \sin t, 2)$$

$$T'(\pi/2) = -\tfrac{1}{2}(1+\pi^2)^{-3/2} \cdot 4\pi\,(-1, 0, \pi) + \frac{1}{\sqrt{1+\pi^2}}(0, 1, 2)$$

$$= \left(\frac{2}{(1+\pi^2)^{3/2}}, \frac{1}{\sqrt{1+\pi^3}}, \frac{-2\pi^2}{(1+\pi^2)^{3/2}} + \frac{2}{\sqrt{1+\pi^2}}\right)$$

so $$N(\pi/2) = \frac{T'(\pi/2)}{|T'(\pi/2)|} = \frac{(\quad '' \quad)}{\sqrt{\frac{4}{(1+\pi^2)^3} + \frac{1}{1+\pi^2} + \left(\frac{-2\pi^2}{(1+\pi^2)^{3/2}} + \frac{2}{\sqrt{1+\pi^2}}\right)^2}}$$

b. $$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|(2\cos t + 2t\sin t, -2t\cos t + 2\sin t, 1)|}{(1+4t^2)^{3/2}} = \frac{\sqrt{(2t\sin t + 2\cos t)^2 + (2\sin t - 2t\cos t)^2 + 1}}{(1+4t^2)^{3/2}}$$

c. $$\int_1^3 \sqrt{1+4t^2}\, dt$$

8. **(5 points)** Find the parametric equation for the tangent line to the curve $x = 1 + 2\sqrt{t}$, $y = t^3 - t$, $z = t^3 + t$ at the point $(3, 0, 2)$. $\leftarrow t=1$

$\vec{r}'(t) = (\frac{1}{\sqrt{t}}, 3t^2-1, 3t^2+1)$

$\vec{r}'(1) = (1, 2, 4)$

$\boxed{(x,y,z) = (3,0,2) + t(1,2,4)}$

9. **(10 points)** At what point to the curves $\vec{r}_1(t) = (t, 1-t, 3+t^2)$ and $\vec{r}_2(t) = (3-t, t-2, t^2)$ intersect? What is the cosine of the angle between them at the point of intersection?

1. $t = 3-s$, $1-t = s-2$, $3+t^2 = s^2$

$\hookrightarrow t=1 \quad s=2$

$\boxed{(1, 0, 4)}$

2. $r_1'(t) = (1, -1, 2t)$ intersect at $r_1(1) = r_2(2)$

$r_2'(t) = (-1, 1, 2t)$

$r_1'(1) = (1, -1, 2)$

$r_2'(2) = (-1, 1, 4)$

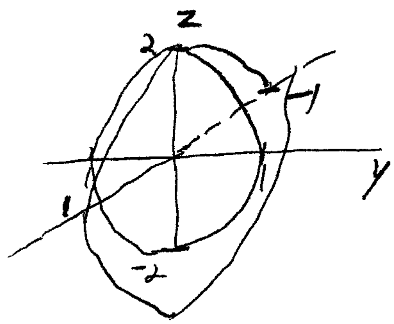
$\cos\theta = \frac{(1,-1,2)\cdot(-1,1,4)}{|(1,-1,2)|\,|(-1,1,4)|} = \frac{6}{\sqrt{6}\sqrt{18}} = \frac{6}{\sqrt{108}}$

10. **(6 points)** Describe a method for determining whether four points P, Q, R and S lie in the same plane.

They do iff $\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = 0$

11. **(10 points)**

a. Neatly sketch the graph of $x^2 + y^2 + z^2/4 = 1$, labeling all intercepts.



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b. Neatly sketch the graph of $z = y^2$ in $\mathbb{R}^3$.