

Name: SOLUTIONS

Math 2950- Midterm Exam #1 - September 27, 2004

1. (10 points) Find the equation of the plane perpendicular to the line

$$(x, y, z) = (1, -2, 3) + t(1, -2, 1)$$

and passing through the point  $(2, 4, -1)$ .

We know the normal vector is  $(1, -2, 1)$

$$\langle x-2, y-4, z+1 \rangle \cdot \langle 1, -2, 1 \rangle = 0$$

or expanded out

$$x - 2y + z = -7$$

2. (15 points) Let  $\vec{u} = (1, -1, 3)$ ,  $\vec{v} = (1, 1, 2)$ . Calculate  $\vec{u} \cdot \vec{v}$  and  $\vec{u} \times \vec{v}$ . Then find the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} \cdot \vec{v} = 1 - 1 + 6 = 6$$

$$\vec{u} \times \vec{v} = (-2 - 3, -2 + 3, 1 - -1) = (-5, 1, 2)$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{6}{\sqrt{11} \sqrt{6}}$$

$$\theta = \cos^{-1} \left( \frac{6}{\sqrt{66}} \right)$$

3. (10 points). Let  $\vec{u}$  be a differentiable vector function and  $f$  a real valued function. Then the chain rule says:

$$\frac{d}{dt} [\vec{u}(f(t))] = ?$$

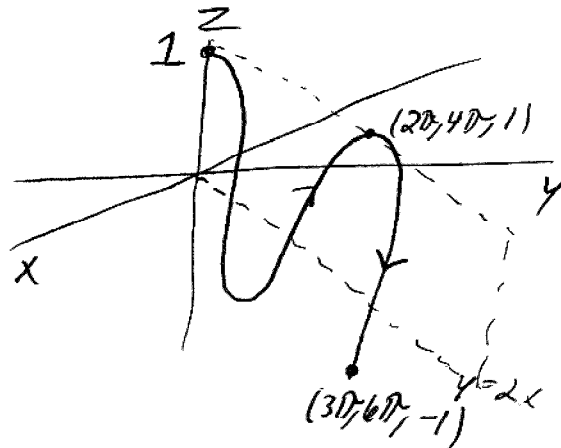
$$\vec{u}'(f(t)) f'(t)$$

4. (10 points) Sketch the space curve

$$\vec{r}(t) = (t, 2t, \cos(t))$$

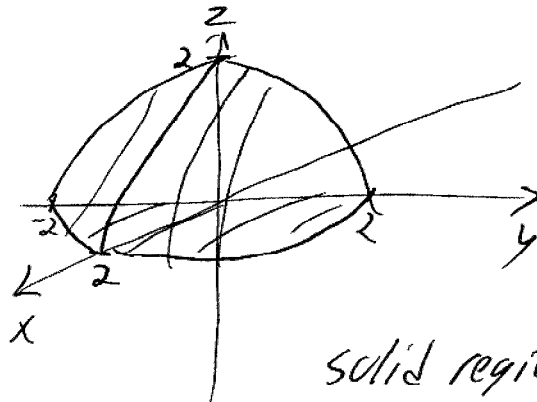
for the interval  $0 \leq t \leq 3\pi$ , indicating with an arrow the direction of increasing  $t$ .

*Notice the curve lies on the plane  $y=2x$ .*



5. (10 points) Sketch the region given by the spherical coordinate inequalities:

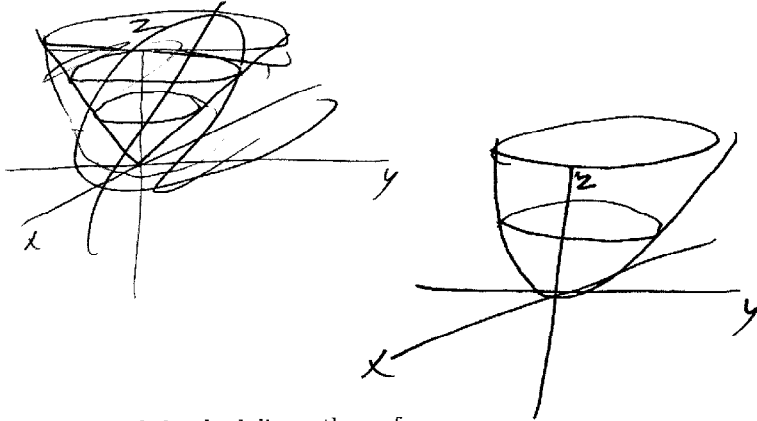
$$\begin{aligned} 0 &\leq \phi \leq \pi/2 \\ -\pi/2 &\leq \theta \leq \pi/2 \\ 0 &\leq \rho \leq 2. \end{aligned}$$



*solid region,  $1/2$  of the upper hemisphere.*

6. (20 points)

a. Sketch the surface given by  $z = x^2 + y^2$ .



b. Verify that the two space curves below both lie on the surface:

$$\vec{r}(t) = (\cos(t), \sin(t), 1) \text{ and } \vec{x}(s) = (s, 0, s^2)$$

$$1 = \cos^2 t + \sin^2 t \text{ so } \vec{r}(t) \text{ is on the surface}$$

$$s^2 = s^2 + 0^2 \text{ so } \vec{x}(s) \text{ is on the surface}$$

c. Verify that the point  $(1, 0, 1)$  lies on both curves. (i.e. find a  $t_0$  such that  $(1, 0, 1) = \vec{r}(t_0)$  and a  $s_0$  such that  $(1, 0, 1) = \vec{x}(s_0)$ .)

$$(1, 0, 1) = \vec{r}(0) \text{ so } (1, 0, 1) \text{ is on } \vec{r}(t)$$

$$(1, 0, 1) = \vec{x}(1) \text{ so } (1, 0, 1) \text{ is on } \vec{x}(s)$$

- d. Find the equation for the tangent line to the curve  $\vec{r}(t)$  at the point  $(1, 0, 1)$ .

$$(x, y, z) = (1, 0, 1) + t(0, 1, 0)$$

$$\vec{r}'(t) = (-\sin t, \cos t, 0)$$

$$\vec{r}'(0) = (0, 1, 0)$$

- e. Find the equation for the tangent line to the curve  $\vec{x}(s)$  at the point  $(1, 0, 1)$ .

$$\vec{x}'(s) = (1, 0, 2s)$$

$$\vec{x}'(1) = (1, 0, 2)$$

$$(x, y, z) = (1, 0, 1) + t(1, 0, 2)$$

- f. Find the equation of the plane passing through the point  $(1, 0, 1)$  and containing both tangent lines above. This is called the *tangent plane* to the surface at the point  $(1, 0, 1)$ .

The normal vector is  $\perp$  to both direction vectors for the lines. Thus

$$\vec{n} = (1, 0, 2) \times (0, 1, 0) = (-2, 0, 1)$$

$$(x-1, y, z-1) \cdot (-2, 0, 1) = 0$$

or

$$-2x + z = -1$$

7. (15 points) a. Find the velocity, acceleration, and speed of a particle with position function given by:

$$r(t) = (t^2 + 1, \cos(t), t)$$

$$\vec{v}(t) = (2t, -\sin t, 1)$$

$$\vec{a}(t) = (2, -\cos t, 0)$$

$$\text{speed} = |\vec{v}(t)| = \sqrt{4t^2 + \sin^2 t + 1}$$

b. Express the arc length of the curve above from  $t = 1$  to  $t = 2$  as a definite integral (do not try to evaluate the integral!).

$$\int_1^2 \sqrt{4t^2 + \sin^2 t + 1} dt$$

8. (10 points) Find the work by a force  $\vec{F} = (1, -1, 1)$  moving an object from the point  $(1, 1, 1)$  to the point  $(4, 2, 1)$ .

$$\vec{D} = (3, 1, 0)$$

$$W = \vec{F} \cdot \vec{D} = 3 - 1 + 0 = \boxed{2}$$