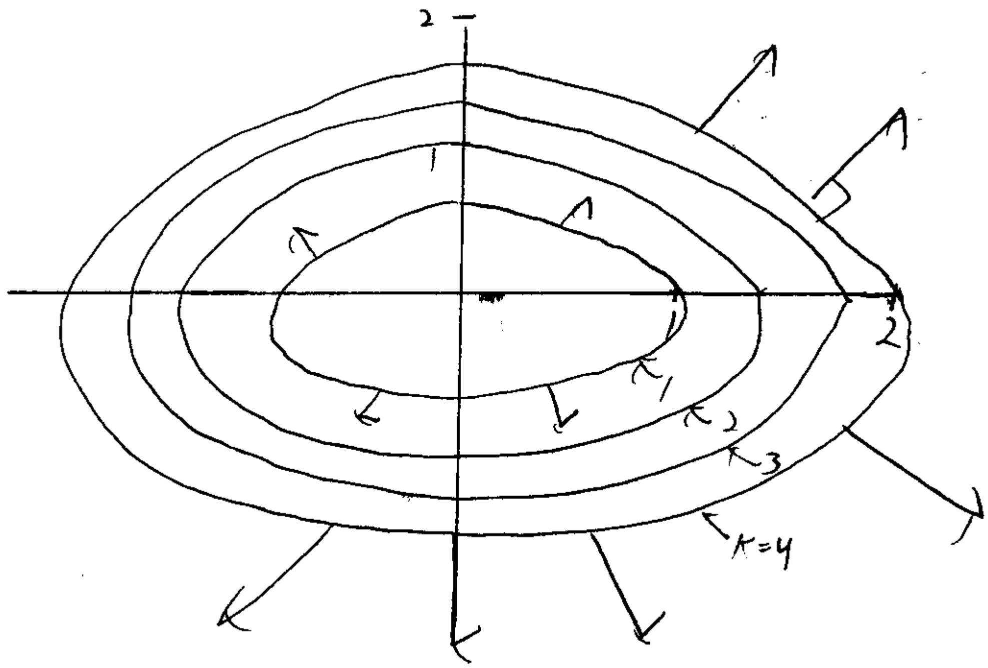
Name: SOLUTIONS

Math 241WW- Final Exam - December 12, 2008

Instructions: The exam is worth 150 points. Calculators are not permitted. You are permitted two index cards, each no larger than 3×5, with whatever you like written on them.

1. **(10 points)** Let $f(x,y) = x^2 + 2y^2$. Sketch level curves $f(x,y) = k$ for $k = 1, 2, 3, 4$ and on the same picture sketch the vector field ∇f.



Recall ∇f is $\perp$ to level curves

b. Find the direction and the rate of maximum increase for $f(x,y)$ at the point $(1,3)$.

$\nabla f = (2x, 4y)$

$\nabla f(1,3) = (2,12)$ ← Direction

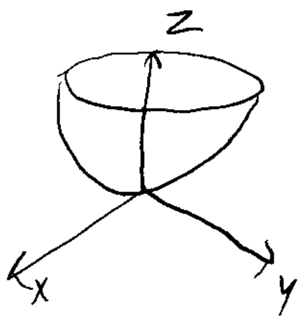
$\sqrt{2^2+12^2}$ ← rate

2. **(15 points)** Let S be the surface $z = x^2 + y^2$ lying below the plane $z = 2$.

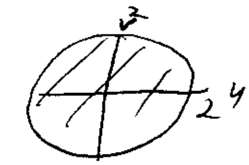
a. Neatly sketch the surface.
b. Parameterize the surface as $\vec{r}(u, v)$ with $(u, v) \in D$. Be sure to describe the region D.
c. Let $\vec{F}(x, y, z) = (z, x, y)$. Use Stokes' theorem to evaluate:

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS.$$

a.



b. $\vec{r}(u,v) = (u, v, u^2+v^2)$ $(u,v) \in$



c. Boundary curve C: $\vec{r}(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t, 2)$ $0 \le t \le 2\pi$

$\vec{r}'(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0)$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^{2\pi} (2, \sqrt{2}\cos t, \sqrt{2}\sin t) \cdot (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0)$$

$$= \int_0^{2\pi} -2\sqrt{2}\sin t + 2\cos^2 t\, dt$$

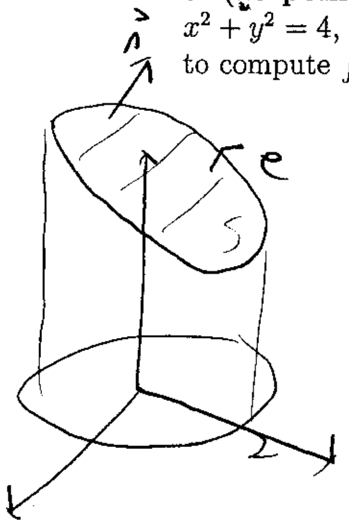
$$= \int_0^{2\pi} -2\sqrt{2}\sin t\, dt + 2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= 0 + 2\left(\frac{t}{2} + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= 2 \cdot \pi$$

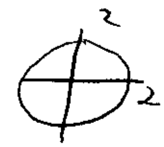
$$= 2\pi$$

3. (**15 points**) Let C be the curve of intersection of the plane $y + z = 7$ and the cylinder $x^2 + y^2 = 4$, oriented to be counterclockwise when viewed from above. Use Stokes' theorem to compute $\int_C \vec{F} \cdot d\vec{r}$ where $F(x, y, z) = (-y^2, x, z^2)$.



parametrize S as

$\vec{r}(u,v) = (u, v, 7-v)$ $\quad (u,v) \in$



Stokes: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}$

$\vec{r}_u = (1,0,0)$ $\quad \vec{r}_v = (0,1,-1)$ $\quad \vec{r}_u \times \vec{r}_v = (0,1,1)$ points up!

$\text{curl}\,\vec{F} = (0, 0, 1+2y)$

$\iint_D (0,0,1+2v) \cdot (0,0,1) = \iint_D 1+2v$ $\quad$ Let $u = r\cos\theta$ $\; v = r\sin\theta$

$= \int_0^2 \int_0^{2\pi} (1 + 2r\sin\theta) r \, dr \, d\theta$

$= \int_0^2 r\theta - 2r^2\cos\theta \Big|_0^{2\pi} dr$

$= \int_0^2 2\pi r \, dr = \pi r^2 \Big|_0^2 = 4\pi$

4. (**15 points**) Let S be the cylinder (including top and bottom) $x^2 + y^2 = 1, 0 \le z \le 1$, with outward pointing normal. Let $\vec{F}(x,y,z) = (x^3, y^3, e^{-z})$. Use the divergence theorem to compute

$$\int\int_S \vec{F} \cdot \vec{n}\, dS. \quad = \iiint_E \operatorname{div} F$$

$E: 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1$ in cyl coor

$$\operatorname{div} F = 3x^2 + 3y^2 - e^{-z} = 3r^2 - e^{-z}$$

$$\int_0^1 \int_0^{2\pi} \int_0^1 (3r^2 - e^{-z}) r \, dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} 3r^3 z + re^{-z} \Big|_{z=0}^{1} d\theta\, d$$

$$= \int_0^1 \int_0^{2\pi} 3r^3 + \tfrac{1}{e} r - r \, d\theta\, dr$$

$$= \int_0^1 2\pi \left(3r^3 + \tfrac{1}{e} r - r\right) dr$$

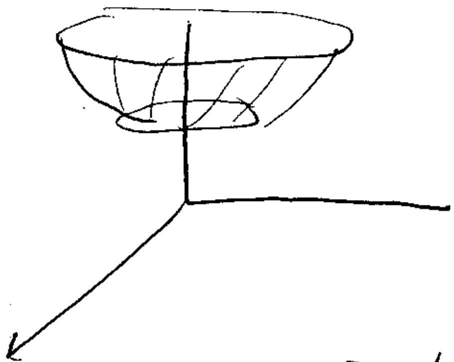
$$= 2\pi \cdot \left(\tfrac{3}{4} r^4 + \tfrac{1}{2e} r^2 - \tfrac{r^2}{2}\right)\Big|_0^1$$

$$= 2\pi \left(\tfrac{3}{4} + \tfrac{1}{2e} - \tfrac{1}{2}\right)$$

$$= 2\pi \left(\tfrac{1}{4} + \tfrac{1}{2e}\right)$$

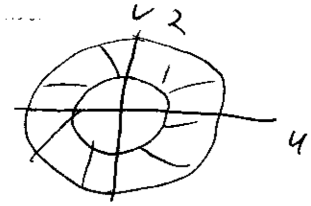
$$= \boxed{\frac{\pi}{2} + \pi / e}$$

5. (**10 points**) Let S be the portion of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 1$ and $z = 4$. Write down *but do not evaluate* an integral which gives the surface area of S.



$S: \vec{r}(u,v) = (u, v, u^2 + v^2)$

$(u,v) \in$



$r_u = (1, 0, 2u) \quad r_v = (0, 1, 2v)$

$r_u \times r_v = (-2u, -2v, 1)$

$|r_u \times r_v| = \sqrt{1 + 4u^2 + 4v^2}$

$A(S) = \iint_D \sqrt{1 + 4u^2 + 4v^2}$

$= \int_1^2 \int_0^{2\pi} \sqrt{1 + 4r^2}\, r \, d\theta \, dr$

6. (**10 points**) Consider the surface parameterized by $\vec{r}(u,v) = (u^2+v^2, uv, uv^2)$. Find the tangent plane to this surface at the point where $u=1, v=2$. Give the equation of the tangent plane in two forms, first parameterize it. Second, give it in $ax+by+cz=d$ form.

$\vec{r}_u = (2u, v, v^2)$ $\vec{r}_v = (2v, u, 2uv)$

At $1=u$, $2=v$ $\vec{r}_u = (2,2,4)$ $\vec{r}_v = (4,1,4)$

$r_u \times r_v = (4, 8, -6)$ use as $\vec{n}$ point is $(5,2,4)$

$$\boxed{\begin{gathered} 4x+8y-6z = 12 \\ r(s,t) = (5,2,4) + s(2,2,4) + t(4,1,4) \quad -\infty < s,t < \infty \end{gathered}}$$

7. (10 points) Use Green's theorem to calculate the work done by the force field $\vec{F}(x,y) = (x^2 - y^3, x^3 + y^2)$ moving a particle around the unit circle $x^2 + y^2 = 1$ in the *clockwise direction.*

Green: $\oint_C \vec{F} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$= \iint_D 3x^2 + 3y^2$$

$$= \int_0^{2\pi} \int_0^1 3r^2 r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} r^4 \Big|_0^1 = 2\pi \cdot \frac{3}{4} = 3\pi/2$$

but we want clockwise so

$-3\pi/2$

8. **(10 points)** Let $f(x, y, z) = x^2y + y^2z + xyz$.

a. Calculate ∇f and $\text{curl}(\nabla f)$.

b. Calculate the line integral: $\int_C \nabla f \cdot d\vec{r}$ where C is the top half of the ellipse $3x^2 + 7y^2 = 12$ traversed from $(-2, 0, 0)$ to $(2, 0, 0)$ in xy plane.

a. $\nabla f = (2xy + yz,\ x^2 + 2yz + xz,\ y^2 + xy)$

$\text{curl}(\nabla f) = (0, 0, 0)$

b. $= f(2,0,0) - f(-2,0,0) = 0 - 0 = 0$

9. **(10 points)** Let $f(x,y) = x^4 + y^2 - 8x^2 - 6y + 16$. Find and classify all critical points of $f(x,y)$ as local max, local min or saddle points.

$\nabla f = (4x^3 - 16x,\ 2y - 6)$

$4x^3 - 16x = 0 \qquad 2y - 6 = 0$

$x = 0, 2, -2 \qquad y = 3$

C.P. $(0,3)$

$(2,3)$

$(-2,3)$

$D = f_{xx} f_{yy} - f_{xy}^2 = (12x^2 - 16)2 - 0 = 24x^2 - 32$

$\boxed{\begin{array}{llll} (0,3) & D = -32 & & \text{saddle} \\ (2,3) & D = 64 & f_{xx} > 0 & \text{local min} \\ (-2,3) & \text{"} & \text{"} & \text{local min} \end{array}}$

10. (10 points) Suppose $w = x^2 e^{yz}$ and $(x, y, z) = (st + u^2, s^2ut, s + tu)$. Find $\frac{\partial w}{\partial u}$ at the point where $s = 1, t = 2, u = 3$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$$

$$= 2xe^{yz}\cdot 2u + x^2 z e^{yz} s^2 t + y x^2 e^{yz}\cdot t$$

$$x = 11 \quad y = 6 \quad z = 7$$

$$= 22e^{42}\cdot 6 + 121\cdot 7\cdot e^{42}\cdot 2 + 121\cdot 6\cdot e^{42}\cdot 2$$

$$= e^{42}(132 + 121\cdot 14 + 121\cdot 12) = \boxed{3278e^{42}}$$

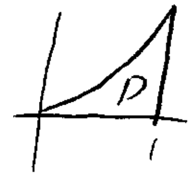
11. (10 points) Find the equation of the plane containing the points $\overset{P}{(1, 2, 3)}, \overset{Q}{(-2, 1, 1)}, \overset{R}{(-3, 1, 2)}$.

$$\vec{PQ} = (-3, -1, -2) \quad \vec{PR} = (-4, -1, -1)$$

$$\vec{PQ} \times \vec{PR} = (-1, 5, -1)$$

$$\boxed{-x + 5y - z = 6}$$

12. (10 points) Let D be the region enclosed by the curves $y = 0$, $y = x^2$ and $x = 1$. Find the average value of $f(x, y) = x \sin y$ over the region D.



$$\text{Area } D = \iint_D 1 = \int_0^1 \int_0^{x^2} 1 \, dy \, dx$$

$$= \int_0^1 x^2 \, dx = 1/3$$

$$\iint_D x \sin y = \int_0^1 \int_0^{x^2} x \sin y \, dy \, dx$$

$$= \int_0^1 -x \cos y \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 -x\cos(x^2) + x \quad dx$$

$$= -\frac{1}{2} \sin(x^2) + x^2/2 \Big|_0^1$$

$$= -1/2 \sin(1) + 1/2$$

$$f_{ave} = \frac{\iint f}{\text{Area}} = \frac{-1/2 \sin(1) + 1/2}{1/3}$$

$$= -3/2 \sin(1) + 3/2$$

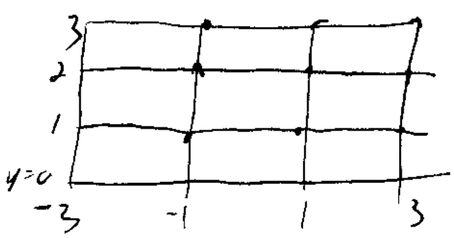
13. (**5 points**) Consider a particle with position at time t given by $\vec{r}(t) = (1+t, t^3, t^2)$. What is the speed of the particle at when $t = 3$?

$r' = (1, 3t^2, 2t)$

$r'(3) = (1, 27, 6)$

$|r'(3)| = \sqrt{1^2 + 27^2 + 36} = \sqrt{764}$

14. (**10 points**) Let R be the rectangle $[-3,3] \times [0,3]$. Estimate $\int\int_R xy^2$ using a Riemann sum with $m = n = 3$ and the upper right corner of each rectangle as your sample point.



Each rect has area 2,

$2(f(-1,1) + f(1,1) + f(3,1) + f(-1,2) + f(1,2) + f(3,2)$

$+ f(-1,3) + f(1,3) + f(3,3))$

$= 2(-1+1+3-4+4+12+-9+9+27)$

$= 2(42) = 84$