

Name: SOLUTIONS

Math 241- Midterm Exam #3 - November 13, 2008

Instructions: You are allowed a single 3" by 5" index card but no other notes, books, or calculators.

1. (15 points) Use spherical coordinates to evaluate:

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2} dz dx dy$$

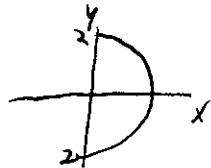
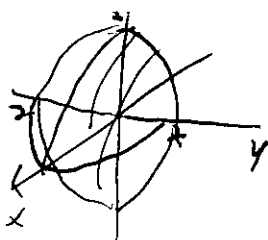
Region

"Front" half of sphere radius 2.

$$0 \leq \rho \leq 2$$

$$-\pi/2 \leq \theta \leq \pi/2$$

$$0 \leq \phi \leq \pi/2$$



$$\int_0^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \rho \cdot \rho^2 \sin \phi d\phi d\theta d\rho$$

$$= \int_0^2 \rho^3 d\rho \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\pi/2} \sin \phi d\phi$$

$$= \frac{\rho^4}{4} \Big|_0^2 \cdot \theta \Big|_{-\pi/2}^{\pi/2} \cdot (-\cos \phi) \Big|_0^{\pi/2}$$

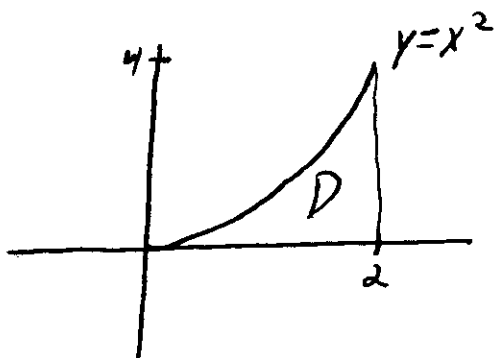
$$= 8 \cdot \pi \cdot (0 - (-1)) =$$

~~8\pi~~

16\pi

2. (10 points) Draw a sketch of the region  $D$  over which the iterated integral is being evaluated, then express it as an iterated integral in which the order of integration is reversed:

$$\int_0^2 \int_0^{x^2} f(x, y) dy dx.$$



$$0 \leq y \leq 4$$
$$\sqrt{y} \leq x \leq 2$$

$$\int_0^4 \int_{\sqrt{y}}^2 f(x, y) dx dy$$

3. (15 points) Find the work done by a force  $\vec{F}(x, y) = (2 - y, x)$  in moving a particle along one arch of the cycloid given by  $\vec{r}(t) = (t - \sin t, 1 - \cos t)$  for  $0 \leq t \leq 2\pi$ .

$$\vec{r}'(t) = (1 - \cos t, \sin t)$$

$$\vec{F}(\vec{r}(t)) = (1 + \cos t, t - \sin t)$$

$$\begin{aligned}\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= (1 + \cos t)(1 - \cos t) + (t - \sin t)\sin t \\ &= 1 - \cos^2 t + t \sin t - \sin^2 t \\ &= 1 - (\sin^2 t + \cos^2 t) + t \sin t \\ &= t \sin t\end{aligned}$$

$$\begin{aligned}\text{Work} &= \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_0^{2\pi} t \sin t dt \\ &= -t \cos t + \sin t \Big|_0^{2\pi} \\ &= -2\pi + 0 - (-0) \\ &= \boxed{-2\pi}\end{aligned}$$

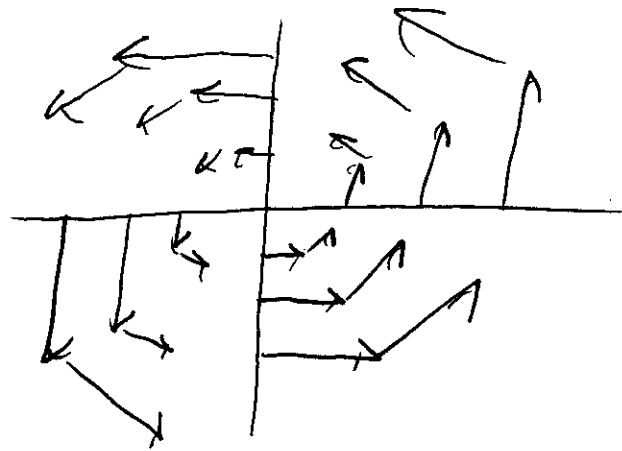
4. (5 points) A vector field  $\vec{F}$  is *conservative* if ...

there is a function  $f$  with

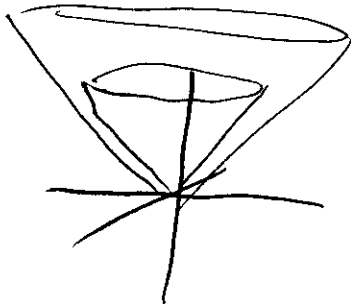
$$\nabla f = \vec{F}$$

5. (10 points) Neatly sketch the vector field below, be sure to sketch enough vectors so the behavior of the vector field is clear.

$$\vec{F}(x, y) = (-y, x)$$



6. (10 points) Find the volume of the part of the ball  $\rho \leq 5$  that lies between the cones  $\phi = \pi/4$  and  $\phi = \pi/3$ .



$$V = \int_0^5 \int_{\pi/4}^{\pi/3} \int_0^{2\pi} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho$$

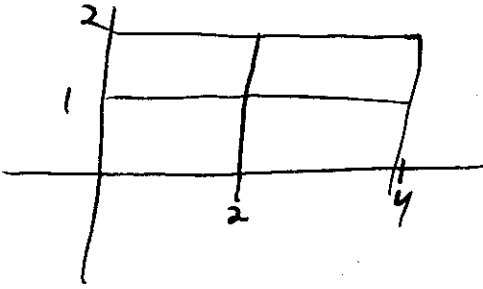
$$= \int_0^5 \rho^2 \, d\rho \int_{\pi/4}^{\pi/3} \sin \phi \, d\phi \int_0^{2\pi} 1 \, d\theta$$

$$= \left. \frac{\rho^3}{3} \right|_0^5 \left( -\cos \phi \right)_{\pi/4}^{\pi/3} \theta \Big|_0^{2\pi}$$

$$= \frac{125}{3} \cdot (-\cos \pi/3 + \cos \pi/4) (2\pi)$$

$$= \frac{250\pi}{3} \left( -1/2 + \frac{\sqrt{2}}{2} \right)$$

7. (10 points) Let  $R$  be the rectangle  $[0, 4] \times [0, 2]$ . Estimate  $\iint_R x^2 y$  using a Riemann sum with  $m = n = 2$  and the upper right corner of each rectangle as your sample point.

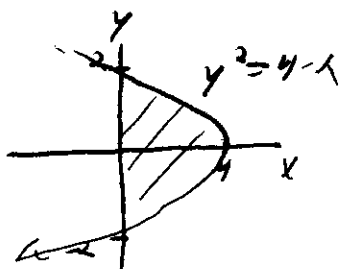


$$\Delta x \Delta y = 2$$

$$\iint_R x^2 y \approx 2 \cdot [f(2,1) + f(4,1) + f(2,2) + f(4,2)]$$

$$= 2(4 + 16 + 8 + 32) = \boxed{120}$$

8. (15 points) Find the volume of the region  $E \subset \mathbb{R}^3$  bounded below by the  $xy$ -plane, above by the plane  $z = x$ , and by the parabolic cylinder  $y^2 = 4 - x$ .



$$0 \leq x \leq 4 - y^2$$

$$-2 \leq y \leq 2$$

Volume =

$$\int_{-2}^2 \int_0^{4-y^2} \int_0^x 1 \, dz \, dx \, dy$$

$$= \int_{-2}^2 \int_0^{4-y^2} x \, dx \, dy$$

$$= \int_{-2}^2 \frac{x^2}{2} \Big|_0^{4-y^2} \, dy$$

$$= \int_{-2}^2 \frac{(4-y^2)^2}{2} \, dy$$

$$= \int_{-2}^2 \left( \frac{1}{2}y^4 - 8y^2 + 16 \right) \, dy$$

$$= \left. \frac{1}{10}y^5 - \frac{8}{3}y^3 + 16y \right|_{-2}^2$$

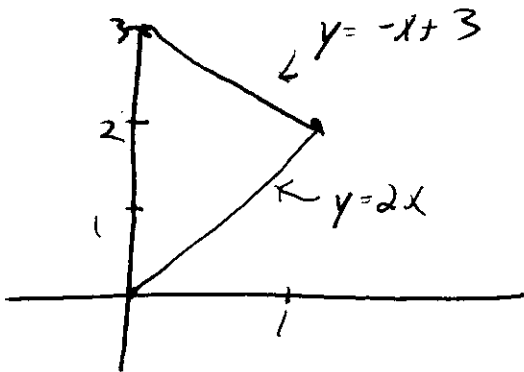
$$= \left( \frac{32}{10} - \frac{64}{3} + 32 \right) - \left( -\frac{32}{10} + \frac{64}{3} - 32 \right)$$

$$= 2 \cdot \left( 32 + \frac{32}{10} - \frac{64}{3} \right)$$

9. (10 points) Evaluate the double integral:

$$\iint_D xy \, dA$$

where  $D$  is the triangular region with vertices  $(0,0)$ ,  $(1,2)$  and  $(0,3)$ .



$$0 \leq x \leq 1$$

$$2x \leq y \leq 3-x$$

$$\int_0^1 \int_{2x}^{3-x} xy \, dy \, dx$$

$$= \int_0^1 \frac{xy^2}{2} \Big|_{y=2x}^{y=3-x} dx$$

$$= \int_0^1 \frac{x}{2} ((3-x)^2 - (2x)^2) dx$$

$$= \int_0^1 \frac{x}{2} (x^2 - 6x + 9 - 4x^2) dx$$

$$= \int_0^1 \frac{x^3}{2} - 3x^2 + \frac{9}{2}x - 2x^3 dx$$

$$= \frac{x^4}{8} - x^3 + \frac{9}{2}x - \frac{1}{2}x^4 \Big|_0^1$$

$$= \frac{1}{8} - 1 + \frac{9}{2} - \frac{1}{2}$$

$$= \frac{1}{8} + 3 =$$

$$\frac{25}{8}$$