Lecture 9

Review Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.
Then for $f(\vec{x}): \mathbb{R}^n \to \mathbb{R}$ say

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \quad \text{if}$$

For all $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\text{If } 0 < |\vec{x} - \vec{a}| < \delta \text{ then } |f(x) - L| < \varepsilon$$

magnitude of vector $\vec{x} - \vec{a}$

absolute value

Rmk Can generalize even more, $f: \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{L}$.

Ex $f(x,y,z) = \left(\frac{x^2+y}{2}, \sin(xy) \right)$

Properties of Limits

1. If limit $= L$ then it $= L$ approaching $\vec{a}$ along any curve. Thus two curves w/ different limits mean limit DNE.

2. All old limit rules still hold, e.g.

- $\lim_{\vec{x} \to \vec{a}} (f(\vec{x}) + g(\vec{x})) = \lim_{\vec{x} \to \vec{a}} f(\vec{x}) + \lim_{\vec{x} \to \vec{a}} g(x)$ if both exist

- $\lim_{\vec{x} \to \vec{a}} (c\vec{f}(\vec{x})) = c \lim_{\vec{x} \to \vec{a}} f(\vec{x})$

- Squeeze Thm: If $g(\vec{x}) \le h(\vec{x}) \le f(\vec{x})$ for all $\vec{x}$ in some ball around $\vec{a}$ and $\lim_{\vec{x} \to \vec{a}} g(x) = \lim_{\vec{x} \to \vec{a}} f(x) = L$

  Then $\lim_{\vec{x} \to \vec{a}} h(x) = L$.

Example

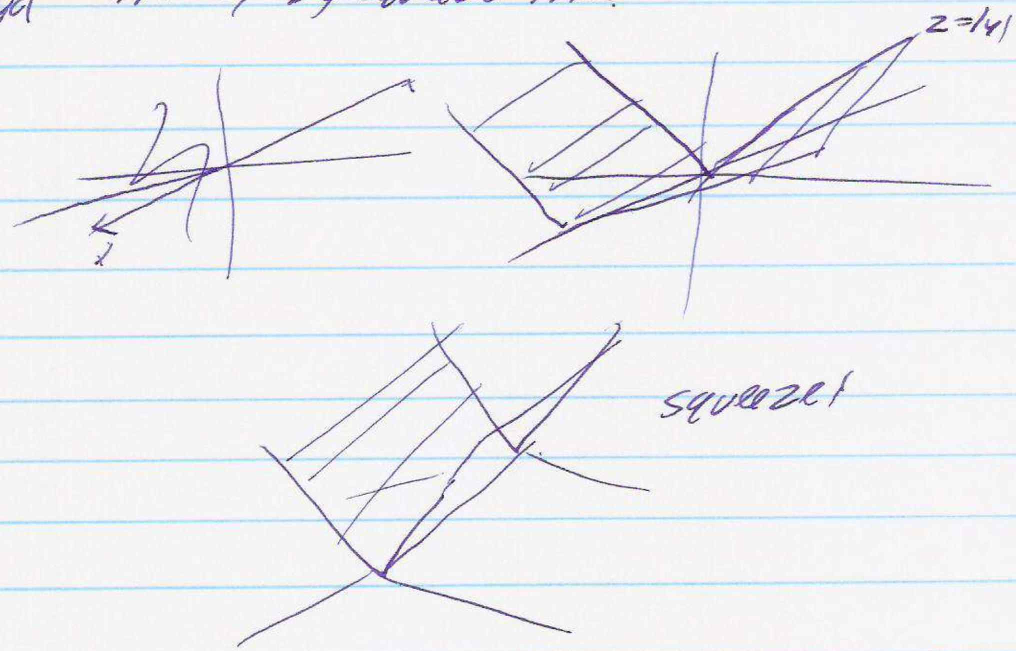
$$f(x,y) = \frac{3x^2y}{x^2+y^2}$$

$$0 < \frac{3x^2}{x^2+y^2} < 3 \quad \text{so}$$

$$0 < \frac{3x^2}{x^2+y^2}|y| < 3|y| \quad \text{Thus}$$

$$-3|y| < f(x,y) < 3|y| \quad \text{and} \quad \lim_{(x,y)\to(0,0)} 3|y| = 0$$
$$= \lim_{(x,y)\to(0,0)} -3|y|$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, by Squeeze Thm.



Review Calc 1

Def $f(x)$ is continuous at $x=a$ if

$$\lim_{x\to a} f(x) = f(a).$$

Rmks • limit exists and..
• $f(a)$ is defined and..
• they are equal!

Limit rules tell us..

<u>Thm</u> Suppose $f(x)$, $g(x)$ are continuous at $x=a$. So are

$$f(x)g(x),\quad f(x)^n,\quad \sqrt[n]{f(x)},\quad f(x)\pm g(x),\quad cf(x),$$

$$\frac{f(x)}{g(x)} \text{ if } g(a)\neq 0 \quad \text{etc.}$$

<u>Example</u>

$$f(x)=\frac{e^x\sin x+x^3}{2x-3} \text{ is continuous except at } x=3/2$$

<u>CALC III - Same!</u>

<u>Def</u> $f(\vec{x})$ is continuous at $\vec{x}=\vec{a}$ if $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x})=f(\vec{a})$

<u>Thm</u> As above.

<u>Example</u>

Find $\lim\limits_{(x,y,z)\to(1,2,-1)} \dfrac{x^2y}{y+2}$

<u>Answer</u> $f(x,y,z)$ is continuous where defined so

$$f(1,2,-1)=\frac{2}{1}=2 \text{ is the limit.}$$

<u>Example</u> Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+1}$

<u>A</u>: Limit $=0$

Example

Find $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$

Answer – Not defined at $(0,0)$ so not continuous

Along $x=y$ $\quad f(x,y) = \frac{x^2\cos x}{4x^2} = \frac{\cos x}{4}$ $\quad$ limit $= \frac{1}{4}$

Along $x=0$ $\quad f(x,y)=0$ $\quad$ limit $= 0$.

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Example

$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ $\quad$ here we have "$\frac{0}{0}$"

try to cancel

$= \lim_{(x,y)\to(0,0)} \frac{\frac{y}{\sqrt{x^2+y^2}}}{x} = \lim \frac{y}{\sqrt{1+\frac{y^2}{x^2}}}$

Notice $\quad 0 < \frac{|y|}{\sqrt{1+\frac{y^2}{x^2}}} < |y|$ so

$-|y| < f(x,y) < |y|$ $\quad$ so limit $= 0$ by S. T.

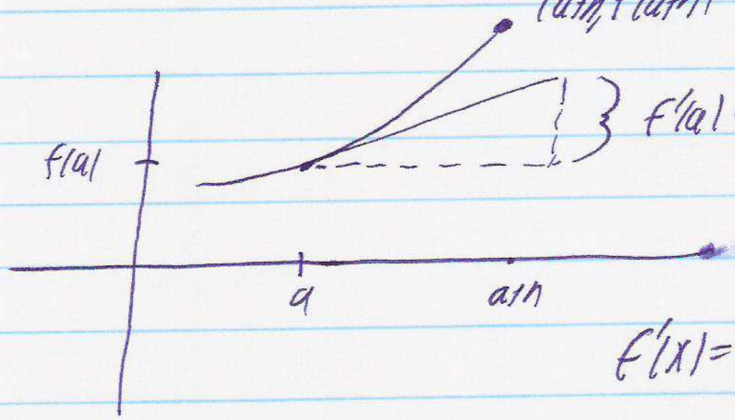
Example Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$

$= \lim_{(r,\theta)\to(0,0)} \frac{e^{-r^2}-1}{r^2} \overset{LHR}{=} \lim_{r\to 0} \frac{-2re^{-r^2}}{2r}$

$= \lim_{r\to 0} \frac{-2e^{-r^2}}{2} = -1$

Partial Derivatives

Review



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Change x from a to $a+h$,

$f(x)$ changes by $\begin{cases} f(a+h) - f(a) & \text{actual amt} \\ f'(a)h & \text{approximate} \end{cases}$

Suppose $f(x,y,z)$ and we change only one variable while holding others constant, how does $f(x,y,z)$ change?

Def $f(x,y)$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Ex $f(x,y) = x\cos(xy)$

$$f_x = \cos(xy) + x\cdot(-y\sin(xy))$$

$$f_y = -x^2\sin(xy)$$