Parameterized curves \( \leftrightarrow \Phi(t) : \mathbb{R} \rightarrow \mathbb{R}^n \)  
Domain \( \subseteq \mathbb{R} \),  
Range \( \subseteq \mathbb{R}^n \)

Chapter 14  
Functions \( \mathbb{R}^n \rightarrow \mathbb{R} \), i.e., many variables, output is real 

**Example**

1. \( f(x,y) = \frac{\ln(x+y)}{x-1} \)  
   Domain: Need \( x+y>0 \)  
   \[ f(x,y) = \frac{\ln(x+y)}{x-1} \]

2. \( f(x,y) = \sqrt{y-x^2} \)  
   Domain

3. Find domain & range of \( f(x,y,z) = \sqrt{1-x^2-y^2-z^2} \)  
   Domain: \( 1-x^2-y^2-z^2 \geq 0 \)  
   \[ 1-x^2-y^2-z^2 \geq 0 \]  
   Inside sphere

   Range \([0,1]\)

**Graphing**

Suppose \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), we usually plot graph as

\( \text{Graph} f = \{ (x,y,z) \in \mathbb{R}^3 | z = f(x,y) \} \)

**Example**

\( f(x) = x^2 \)  
\( \Delta \text{min} (x = 1) \)

\( f(x,y) = x^2 + y^2 \)  
Paraboloid, \( \mathbb{R}^2 \)
Example
\[ f(x,y) = x^2 - y^2 \]

Graph

Example
\[ f(x,y) = ax + by + c \] called a linear function of \( ax + by \)

Graph is plane \( z = ax + by + c \).

Level Curves

Graph \( z = f(x,y) \) is just one way to represent a function. Another is to sketch level curves \( f(x,y) = k \) for various choices of \( k \).

Example
\[ f(x,y) = x^2 + y^2 \]

Sometimes called "contour map" or "isothems" or "isosours".
Example

Let \( f(x,y) = x^2 - y \). Sketch level curves.

\[ k = 0 \quad y = x^2 \]

Example

\( f(x,y) = e^{\frac{y}{x}} \)

\[ k = e^{\frac{y}{l_2}} \]

\[ l_2 = \frac{y}{l_2} \quad y = x/l_2 \]

Computers are very helpful for plotting!

Example

\( f(x,y,z) = x^2 + y^2 + \frac{z^2}{4} \)

Sketch level surfaces.

Level surfaces are ellipsoids long in the z direction.

Note: To graph this function would require 4-dimensions, i.e.,

\( W = x^2 + y^2 + \frac{z^2}{4} \)

Example

\( f(x,y) = \arcsin (x^2 y^2 - 2) \). Find and sketch the domain.
Limits

\[ f(x, y) \rightarrow L \]

Pick \( \varepsilon > 0 \) (think small).

There is a \( \delta > 0 \) so the disk \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \) maps into \( (L - \varepsilon, L + \varepsilon) \).

\( \delta \) on \( f(a, b) \) is irrelevant.

Def. Let \( f \) be a function of two variables whose domain \( D \) includes points arbitrarily close to \( (a, b) \). Then

\[ \lim_{(x, y) \to (a, b)} f(x, y) = L \]

For every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ (x, y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon. \]

General def. \( f(x_1, x_2, x_3) \)

\[ \lim_{(x_1, x_2, x_3) \to (a_1, a_2, a_3)} f(x_1, x_2, x_3) = L \]
Useful idea

Suppose \( f(x,y) \to L_1 \) as \((x,y)\to(a,b)\) along path \(C_1\) and

Suppose \( f(x,y) \to L_2 \) along path \(C_2\) with \(L_1 \neq L_2\).

Then \( \lim_{(x,y)\to(a,b)} f(x,y) \) DNE.

Ex

\[
\lim_{(x,y)\to(a,b)} \frac{xy}{x^2+y^2}
\]

Approach on \( y=0, x=0, y=x \)

Ex

\[
F(x,y) = \frac{xy^2}{x^2+y^2}
\]

\[
\lim_{(x,y)\to(a,b)} F(x,y) = ?
\]

- \( y = mx \) all \( m \to 0 \)
- Parameter \( x-y^2 \)

Limits are complicated!