Lecture 7

Review

Suppose $\vec{r}(t)$ is position of a particle at time t. Then

$\vec{v}(t) = \vec{r}\,'(t)$ is velocity, $|\vec{v}(t)|$ is speed

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t)$ is acceleration

Remark Can integrate to go from $\vec{a} \to \vec{v} \to \vec{r}(t)$.

Example $\vec{a}(t) = \vec{i} + 3\vec{j}$ $\vec{v}(0) = \vec{i} + \vec{k}$ $\vec{r}(0) = \vec{i}$ Find $\vec{v}(t)$ & $\vec{r}(t)$

$\vec{a}(t) = (1,3,0)$ $\vec{v}(t) = (t, 3t, 0) + \vec{C}$ $\vec{C} = (1,0,1)$

$\vec{v}(t) = (t+1, 3t, 1)$

$\vec{r}(t) = (\frac{1}{2}t^2 + t, \frac{3}{2}t^2, t) + \vec{C}$ $\vec{C} = (1,0,0)$

$$\boxed{\vec{r}(t) = (\tfrac{1}{2}t^2 + t + 1, \tfrac{3}{2}t^2, t)}$$

Newton's 2nd Law of Motion

$$\vec{F}(t) = m\,\vec{a}(t)$$

Example $\vec{r}(t) = (a\cos bt, a\sin bt)$ Find Force if mass $= m$

$v(t) = (-ab\sin bt, ab\cos bt)$

$\vec{a}(t) = (-ab^2\cos bt, -ab^2\sin bt)$

$\vec{F} = -mb^2\,\vec{r}(t)$ centripetal force

Remark speed $= ab$

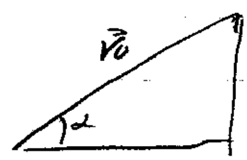
Example

Fire projectile, initial velocity $\vec{V_0}$, angle α.
Assume only force is gravity down.
Find $\vec{r}(t)$, Find α to maximize range

Solution Assume start at origin. Then $\vec{F} = m\vec{a} = -mg\vec{j}$
$= (0, -mg)$

Thus
$$\vec{a}(t) = (0, -g)$$
$$\vec{V}(t) = (0, -gt) + \vec{C} \qquad \vec{C} = V(0) = \vec{V_0}$$
$$\vec{V}(t) = (0, -gt) + \vec{V_0}$$
$$\vec{r}(t) = (0, -\tfrac{1}{2}gt^2) + \vec{V_0}t + \vec{D} \qquad \vec{D} = 0$$

$$\boxed{\vec{r}(t) = -\tfrac{1}{2}gt^2\vec{j} + \vec{V_0}t}$$



Let $V_0 = |\vec{V_0}|$

Thus
$$\vec{r}(t) = ((V_0\cos\alpha)t, (V_0\sin\alpha)t - \tfrac{1}{2}gt^2)$$

Hits ground when $V_0\sin\alpha\, t - \tfrac{1}{2}gt^2 = 0$, i.e. $t = \frac{2V_0\sin\alpha}{g}$

Distance travelled $= \frac{2V_0\sin\alpha}{g} \cdot V_0\cos\alpha = \frac{V_0^2}{g} \cdot 2\sin\alpha\cos\alpha$

$$= \boxed{\frac{V_0^2}{g}\sin(2\alpha)}$$

maximized when $\alpha = \pi/4$

$$\boxed{D = \frac{V_0^2}{g}}$$

Tangential & Normal Components of Accel

Review

Let $s = |\vec{r}\,'(t)|$ = speed. Note • s is a function of t
$= |\vec{v}(t)|$ • book writes v

(*) $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{s}$ is unit tangent vector.

(**) $\kappa = \dfrac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \dfrac{|\vec{T}'(t)|}{s}$ curvature

(***) $\vec{N} = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ unit normal vector.

Analysis:

$\vec{v} = s\,\vec{T}$ by (*)

$\vec{a} = s'\,\vec{T} + s\,\vec{T}'$ apply $\frac{d}{dt}$

$\vec{a} = s'\vec{T} + s\,|\vec{T}'| \cdot \dfrac{\vec{T}'}{|T'|}$

$\vec{a} = s'\vec{T} + s s\kappa\,\vec{N}$ by (**) and (***)

Conclude

$$\boxed{\vec{a} = s'\,\vec{T} + \kappa s^2\,\vec{N}}$$

This gives component of $\vec{a}$ in tangent direction and Normal direction

Observe

1. Constant speed $\longleftrightarrow$ $s' = 0$ $\longleftrightarrow$ accel all in normal direction
2. Accelleration all in osculating plane no $\vec{B}$ component
3. High speed, sharp turn gives lots of Normal accel.

<u>Rmk</u> As before we can get $\vec{a} = a_T \vec{T} + a_N \vec{N}$ with

$$a_T = \frac{\vec{r}'(t) \cdot r''(t)}{|r'(t)|} \qquad a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

Problems

1. $\vec{r}(t) = \langle t\sin t, t\cos t, t^2 \rangle$ Find velocity, accel & speed

$$\vec{v}(t) = \langle \sin t + t\cos t,\ \cos t - t\sin t,\ 2t \rangle$$
$$\vec{a}(t) = \langle \cos t - t\sin t + \cos t,\ -\sin t - t\cos t - \sin t,\ 2 \rangle$$

$$\text{speed} = \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + 4t^2}$$
$$= \sqrt{\sin^2 t + t^2\cos^2 t + \cos^2 t + t^2 \sin^2 t + 4t^2}$$
$$= \sqrt{1 + 5t^2}$$

2. $\vec{r}(t) = (3t - t^3, 3t^2)$ Find tangential & normal comp of accel.

<u>Method 1</u> $\vec{v}(t) = (3 - 3t^2, 6t) = \vec{r}'(t)$ $\sqrt{9t^4 - 18t^2 \cancel{+9+36t^2}\ 36}$

$*$ $r''(t) = (-6t, 6)$

2.. $\vec{r}(t) = (3t - t^3, 3t^2)$

$\vec{r}'(t) = (3 - 3t^2, 6t)$ $\quad s = \sqrt{9t^4 - 18t^2 + 9 + 36t^2}$

$= \sqrt{9t^4 + 18t^2 + 9} = \sqrt{(3t^2+3)^2}$

$= 3t^2 + 3$

$\vec{a}(t) = r''(t) = (-6t, 6)$

$$a_T = \frac{\vec{r}' \cdot r''}{|r'(t)|} = \frac{-18t + 18t^3 + 36t}{3t^2+3}$$

$$= \frac{18t^3 + 18t}{3t^2+3} = \frac{18t(t^2+1)}{3(t^2+1)} = 6t$$

$$a_N = \frac{|(3-3t^2, 6t, 0) \times (-6t, 6, 0)|}{3t^2+3} = \frac{|(0, 0, 18 - 18t^2 + 36t^2)|}{3t^2+3}$$

$$= \frac{18t^2 + 18}{3t^2+3}$$

$= 6$

Method 2

$a_T = \vec{r}'' \cdot \vec{T}$

$a_N = \vec{r}'' \cdot \vec{N}$