Lecture 6

Review

$\vec{r}(t) = (f(t), g(t), h(t))$ is parameterized space curve (in $\mathbb{R}^3$, other dimensions similarly)

Then $\vec{r}\,'(t) = (f'(t), g'(t), h'(t))$.

- at a specific time $t = t_0$, $\vec{r}\,'(t_0)$ is tangent to curve at $\vec{r}(t_0)$
- When t is time, $\vec{r}\,'(t)$ called velocity vector
- $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|r'(t)|}$ is unit tangent vector
- $|\vec{r}\,'(t)|$ is speed.
- $\vec{r}\,''(t)$, $\vec{r}\,'''(t)$, etc...

Diff Rules Let $\vec{u}$ & $\vec{v}$ be diffble vector functions, c a scalar, f a real valued function. Then

Thm

1. $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}\,'(t) + \vec{v}\,'(t)$

2. $\frac{d}{dt}(c\vec{u}(t)) = c\,\vec{u}\,'(t)$

3. $\frac{d}{dt} f(t)\vec{u}(t) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

4. $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

5. $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

6. $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}\,'(f(t))$

Example

1. $\vec{u}(t) = \langle \sqrt{t}, \cos t, t^2+1 \rangle \quad f(t) = t^3 \quad \vec{u}(f(t)) = \langle \sqrt{t^3}, \cos(t^3), t^6+1 \rangle$

$\vec{u}'(t) = 3t^2 \langle \frac{1}{2\sqrt{t}}, -\sin t, 2t \rangle$

2. $f(t)\,\vec{u}(t) = \langle t^3\sqrt{t}, t^3\cos t, t^3(t^2+1) \rangle$

$\frac{d}{dt}\left(f(t)\,\vec{u}(t)\right) = 3t^2 \langle \sqrt{t}, \cos t, t^2+1 \rangle + t^3 \langle \frac{1}{2\sqrt{t}}, -\sin t, 2t \rangle$

Remark If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$

Then

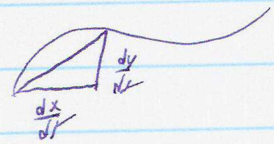
$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

$$= \int_a^b f(t)\,dt\ \vec{i} + \int_a^b g(t)\,dt\ \vec{j} + \int_a^b h(t)\,dt\ \vec{k}$$

Interpretation later.

Arc Length and Curvature

Recall 1. Curve $(x(t), y(t))$ $a \le t \le b$ $\quad A.L. = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

came from 

limits of inscribed segments

2. A.L. of $y = f(x)$ for $a \le x \le b$ $\quad \int_a^b \sqrt{1 + f'(x)^2}\, dx$

Theorem (Arc Length of Parameterized Curve)
The length of $\vec{r}(t)$ for $a \le t \le b$ is

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

Remarks

1. Think: Distance $= \sum$ speed $\cdot$ time $\rightsquigarrow \int$ speed dt

2. This generalizes both previous formulas since we can parameterize $y = f(x)$ by

$$\vec{r}(t) = (t, f(t)) \text{ so}$$
$$\vec{r}\,'(t) = (1, f'(t)) \text{ so}$$
$$|\vec{r}\,'(t)| = \sqrt{1 + f'(t)^2}$$

3. L measures total distance, e.g. $\vec{r}(t) = (\cos t, \sin t)$ $0 \le t \le 4\pi$

$$L = \int_0^{4\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{4\pi} 1\, dt = t\Big|_0^{4\pi} = 4\pi$$

TWICE Around

Ex Find length of $\vec{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$ $-10 \le t \le 10$

$\vec{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle$

$$|\vec{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$$

$$\int_{-10}^{10} \sqrt{29}\, dt = \boxed{20\sqrt{29}}$$

Rmk Most integrals arising in A.L. problems are very difficult, can use computer to approximate.

CURVATURE

Def $\vec{r}(t)$ is smooth if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ $\forall t$

- no sudden change in direction
- no corners

When $\vec{r}(t)$ is smooth then $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is continuous.

Def Given $\vec{r}(t)$ $a \le t \le b$, define:

$$s(t) = \int_a^t |\vec{r}'(u)|\, du = \text{arc length function}$$

= "distance so far"

so $\dfrac{ds}{dt} = |\vec{r}'(t)|$ by FTOC

Def The curvature of a curve is

$$\boxed{\kappa = \left| \frac{d\vec{T}}{ds} \right|} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \quad \text{so} \quad \boxed{\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

Curvature Examples

1. Line $\vec{r}(t) = (x_0+at, y_0+bt, z_0+ct)$

$$T(t) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \qquad T'(t) = 0 \qquad \text{so } \boxed{\kappa = 0}$$

2. Circle radius a. $\vec{r}(t) = (a\cos t, a\sin t)$

$$\vec{r}'(t) = (-a\sin t, a\cos t) \qquad |r'(t)| = a$$

$$T(t) = (-\sin t, \cos t)$$

$$T'(t) = (-\cos t, -\sin t) \qquad |T'(t)| = 1$$

$\boxed{\kappa = 1/a}$ Fundamental example!

Alternate Formula

$$\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$ Proof compute

Useful Lemma

Suppose $|\vec{W}(t)| = 1$. (Example: unit tangent $\vec{T}(t) = \frac{\vec{r}(t)}{|\vec{r}(t)|}$)

Then $\vec{W}(t) \perp \vec{W}'(t)$.

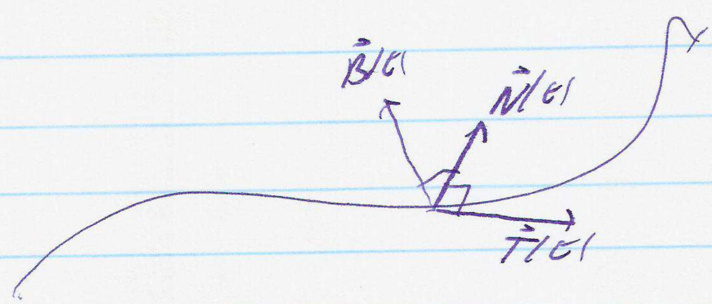
Proof $\vec{W}(t) \cdot \vec{W}(t) = 1$

$$0 = (\vec{W} \cdot W)' = \vec{W} \cdot \vec{W}' + \vec{W}' \cdot \vec{W}$$
$$= 2\vec{W} \cdot \vec{W}'$$
so $\vec{W} \cdot \vec{W}' = 0$.

Corollary $T'(t)$ is $\perp$ to $T(t)$

Def $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|T'(t)|}$ is a unit normal vector.

Def $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ binormal vector.



travelling frame.

$\vec{T}$ & $\vec{N}$ span osculating plane.