Lecture 5

Def A <u>vector valued</u> function is a function w/ domain in $\mathbb{R}$ and range in $\mathbb{R}^n$.

Ex $\vec{r}(t) = (2t+5, t^3, \sin t, t)$ $\quad \vec{r}: \mathbb{R} \to \mathbb{R}^4$ $\quad -\infty < t < \infty$

$\vec{w}(t) = (\cos t, \sin t)$ $\quad 0 \le t \le 2\pi$ $\quad \vec{w}: [0, 2\pi] \to \mathbb{R}^2$

Def Suppose $\vec{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$. Then

$$\lim_{t \to a} \vec{r}(t) = \left(\lim_{t \to a} f_1(t), \ldots, \lim_{t \to a} f_n(t) \right)$$

if each $\lim_{t \to a} f_i(t)$ exists i.e. we take limits one coordinate at a time.

Rmk Can do a "for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$0 < |t - a| < \varepsilon$ implies $|\vec{r}(t) - \vec{L}| < \delta$"

$\uparrow$ vector norm,

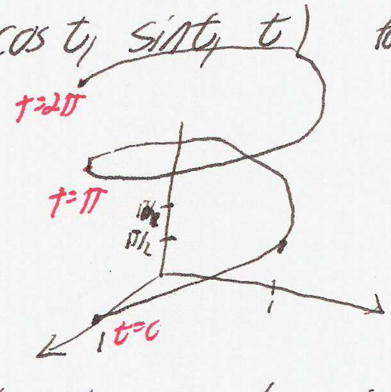
i.e. $\vec{r}(t)$ is in a <u>ball</u> of radius δ around $\vec{L}$.

Def $\vec{r}(t)$ is <u>continuous at a</u> if

$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$, same def as for functions $\mathbb{R} \to \mathbb{R}$

Rmk We can thing of $\vec{r}(t) = (f(t), g(t), h(t))$ as a <u>space curve</u>, as t varies the "particle" traces out curve.

Ex $\vec{r}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$



$t = 0$ $\quad \vec{r}(0) = (1, 0, 0)$

$r(\pi/2) = (0, 1, \pi/2)$

$r(2\pi) = (1, 0, 2\pi)$

* Just sketching the image loses info as to passage of time

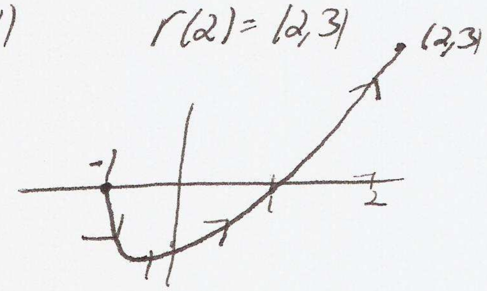
EX $\vec{r}(t) = (3-t,\ t,\ 5+6t) \quad t \in \mathbb{R}$

$= (3,0,5) + t(-1,1,6)$ is a parameterized line

EX $\vec{r}(t) = (t,\ t^2-1) \quad -1 \le t \le 2$

A: Notice every point on this curve lies on parabola $y = x^2 - 1$

$\vec{r}(-1) = (-1,0) \qquad r(2) = (2,3)$



arrow traditionally points in direction of increasing t.

EX Any graph of a function can be parameterized,

i.e. $\quad y = f(x) \longleftrightarrow \vec{r}(t) = (t, f(t))$

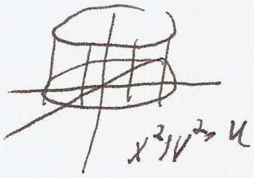
The previous example is a special case

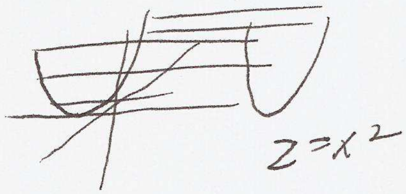
EX $\vec{r}_1(t) = (t^2, 7t-12, t^2) \quad \vec{r}_2(t) = (4t-3, t^2, 5t-6)$

1. Do these particles collide ?

2. Do their paths intersect ?

Ex Find parametric equations for the intersection of $x^2+y^2=4$ and $z=x^2$



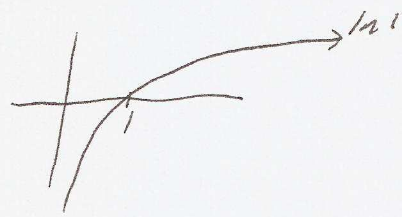


1. We know x,y part is $\cancel{(2\cos t, \sin 2t)}$, $(2\cos t, 2\sin t)$

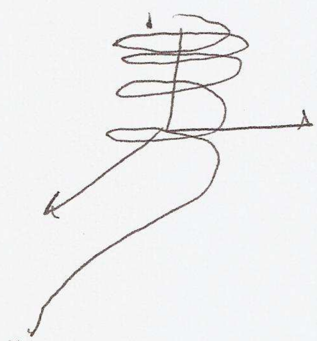
2. Thus $\vec{r}(t) = (2\cos t, 2\sin t, 4\cos^2 t)$

How to sketch? Computers!

Ex Sketch $\vec{r}(t) = (\cos t, \sin t, \ln t)$ $0<t$



A: as $t \to 0^+$, $\ln t \to -\infty$ fast
as $t \to \infty$, $\ln t \to \infty$ very slow!

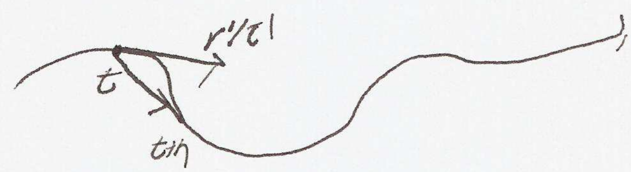


Calculus of Vector Functions

* All as before, just coordwise.

Ex Given $\vec{r}(t)$ define $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$, if it exists

Observe $\vec{r}\,'(t)$ is a vector!



curve in $\mathbb{R}^n$

* Notice that $\vec{r}\,'(t)$ is tangent to the curve, it is called a <u>tangent vector</u>

Thm If $\vec{r}(t) = (f(t), g(t), h(t))$ then $\vec{r}\,'(t) = (f'(t), g'(t), h'(t))$

Def $\vec{T}(t) = \frac{r'(t)}{|r'(t)|}$ is a <u>unit tangent vector</u>.

Example Find tangent line to $\vec{r}(t) = (2\sin t, \cos t, t^2)$

when $t = \pi/2$.

point: $\vec{r}(\pi/2) = (2, 0, \pi^2/4)$

$\vec{r}\,'(t) = (2\cos t, -\sin t, 2t)$

$\vec{r}\,'(\pi/2) = (0, -1, \pi)$ ← tangent vector

Tangent line: $\boxed{(x, y, z) = (2, 0, \pi^2/4) + t(0, -1, \pi)}$

Diff Rules Let $\vec{u}, \vec{v}$ be diff vector functions, c a scalar, f a function

1. $[\vec{u}(t)+\vec{v}(t)]' = \vec{u}'(t)+\vec{v}'(t)$
2. $\frac{d}{dt}(c\,\vec{u}(t)) = c\,u'(t)$
3. $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$
4. $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}(t)\cdot\vec{v}'(t) + \vec{u}'(t)\cdot\vec{v}(t)$
5. $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$
6. Chain Rule $\frac{d}{dt}(\vec{u}(f(t))) = (\vec{u}'(f(t))f'(t)$

Rmk All come from Calc I rules!

EX #5 $\vec{u}(t) = (f(t), g(t), h(t))$

$\vec{v}(t) = (x(t), y(t), z(t))$

$\vec{u}(t)\times\vec{v}(t) = (g(t)z(t)-h(t)y(t),\ h(t)x(t)-f(t)z(t),\ f(t)y(t)-g(t)x(t))$

Now take deriv, apply product rule six times!

EX #6 $\vec{u}(t) = (\sqrt{t}, t^2, e^t)$

$f(t) = \cos t$

$\vec{u}(f(t)) = (\sqrt{\cos t}, \cos^2 t, e^{\cos t})$