Lecture 4

Review $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$ $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

Key Facts

1. $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}, \vec{b}$ in direction given by rt hand rule

2. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ is area of ▱ spanned by $\vec{a}$ & $\vec{b}$

3. In general $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

   Also $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ so $\vec{a} \times \vec{a} = 0$

4. Volume of parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$ is the magnitude of the scalar triple product

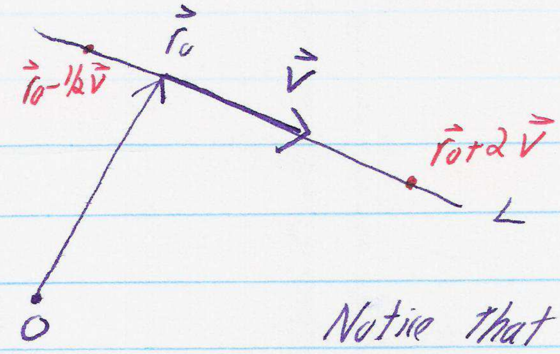
$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

5. Torque $\tau = \vec{r} \times \vec{F}$

Lines and Planes

Recall $y = 5x + 7$ is a line in xy-plane but a plane in $\mathbb{R}^3$, etc..., need new way to represent line in higher dimensions.

LINES



Notice that $\vec{r}_0 +$ any mult of $\vec{v}$ gives the line L.

Thus $\vec{r} = \vec{r}_0 + t\vec{v}$ gives a <u>vector equation</u> of line,
t is a <u>parameter</u>.

If $\vec{r} = (x, y, z)$ $\quad \vec{r}_0 = (x_0, y_0, z_0)$ $\quad \vec{v} = (a, b, c)$ then

$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$ gives 3

scalar equations
$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

<u>Warning</u>:
- Any point (x_0, y_0, z_0) on the line works
- Any $\neq 0$ multiple of $\vec{v}$ works so there are many parameterizations of same line.

Solving for t gives $\boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$ symmetric equations

<u>Example</u>

1. Find parameterization of line through $(-1, 2, 3) = P$ and $(2, 5, 1) = Q$. At what point does it hit the xy plane?

<u>A</u>: Use $P = \vec{r}_0$ $\quad \vec{PQ} = (3, 3, -2)$

$(x, y, z) = (-1, 2, 3) + t(3, 3, -2) = \boxed{(-1 + 3t, 2 + 3t, 3 - 2t)}$

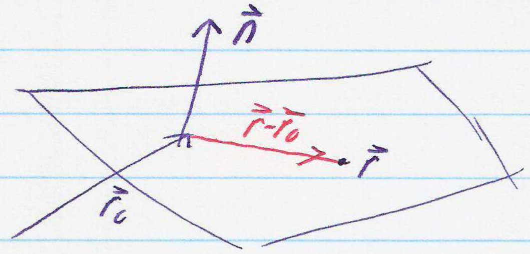
$z = 0$ when $t = 1.5$ so $\boxed{(-3.5, 6.5, 0)}$

Rmk to get a line segment from $\vec{r}_0$ to $\vec{r}_1$ we use

$$\boxed{\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad 0 \le t \le 1}$$

PLANES Line needs only a point + a direction.

Plane needs a point and a normal vector $\vec{n}$ $\perp$ to plane.



Let $\vec{r} = (x,y,z)$. Notice that $\vec{r}$ is in the plane iff $(\vec{r}-r_0) \perp \vec{n}$, i.e. $(\vec{r}-\vec{r}_0)\cdot\vec{n} = 0$.

$$\boxed{\text{vector equation} \quad (\vec{r}-\vec{r}_0)\cdot\vec{n} = 0}$$

Ex $\left((x,y,z)-(1,2,1)\right)\cdot(-3,2,5) = 0$

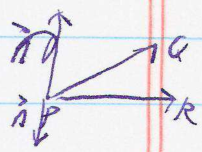
If $\vec{n} = (a,b,c)$ $r_0 = (x_0,y_0,z_0)$ we get

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

scalar eq. of plane through (x_0,y_0,z_0)
w/ normal vector (a,b,c)

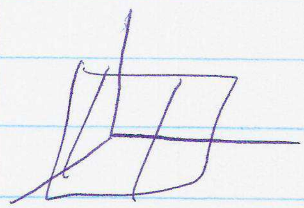
Ex Find eq of plane through $(2,1,-2) = P$, $(1,0,1) = Q$, $(-2,-2,1) = R$
Sketch it

A: $\vec{PQ} = (-1,-1,3)$ $\vec{PR} = (-4,-3,3)$ $\vec{PQ} \times \vec{PR} = (6,-9,-1)$



use $\vec{n} = (6,-9,-1)$ $x_0 = (-1,-1,3)$

$$6(x+1) - 9(y+1) - 1(z-3) = 0$$
$$6x - 9y - z = 0$$



Some tidbits

1. Planes are parallel iff two normal vectors are multiple of each other.
2. If not, the angle between two planes is the angle between the two normal vectors.

Ex Find eq of plane containing $(2,-1,3)$ and parallel to $7x - y + 2z = 3$.

A: $\vec{n} = (7,-1,2)$ so $\big((x,y,z) - (2,-1,3)\big) \cdot (7,-1,2) = 0$

$$(x-2, y+1, z-3) \cdot (7,-1,2) = 0$$

Ex Are $x+y+z=1$ and $x-y+z=1$ $\parallel$ or $\perp$? Find $\measuredangle$ btw them

A $n_1 = (1,1,1)$ $n_2 = (1,-1,1)$ $\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1}{\sqrt{3}\sqrt{3}} = 1/3$

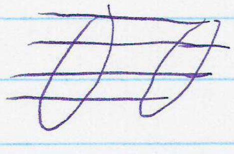
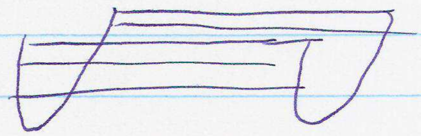
$$\theta \cong 70.5^\circ$$

Ex ~~Find parametric eqs for line through $(0,1,2)$ and $\perp$ to line $x=1+t, y=1-t, z=2t$ and A this line.~~

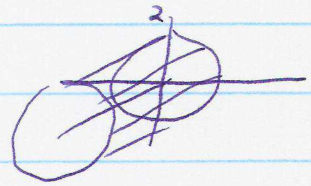
A

Cylinders & Quadrics

Def A *cylinder* is a surface which is all lines // to a given line passing through a given curve.

Ex  Ex 

Ex $y^2+z^2=4$

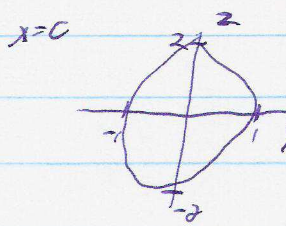


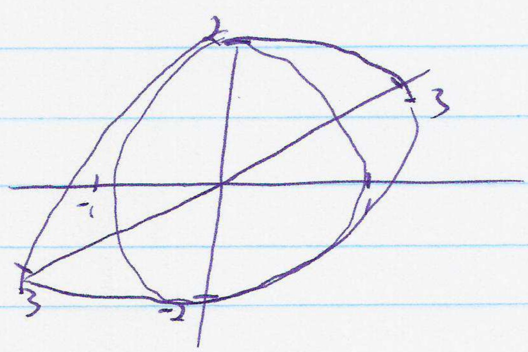
Def A *quadric surface* is graph of degree 2 eq in variables x, y, z, i.e.

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

Rmk To sketch, helpful to look at curves of intersection with 3 coord planes

Ex $\frac{x^2}{9}+y^2+\frac{z^2}{4}=1$





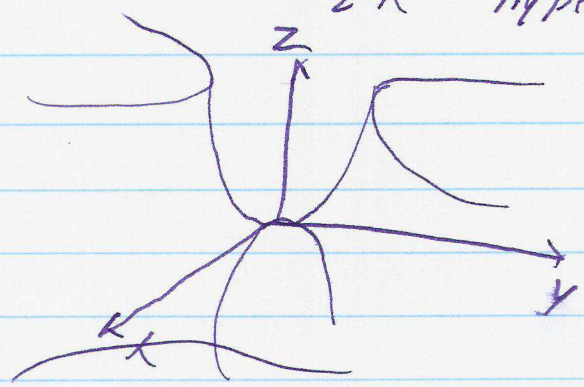
ellipsoid

<u>Ex</u> $z = y^2 - x^2$

$x = k$ get vertical parabolas open up

$y = k$ parabolas open down

$z = k$ hyperbolas



saddle
aka
hyperbolic
paraboloid

$\boxed{\text{Study Table on p. 808}}$

<u>Ex</u> $4y^2 + z^2 - x - 16y - 4z + 20 = 0$ what does it look like?

$4y^2 - 16y \quad + z^2 - 4z = x - 20$

$y^2 - 4y + \frac{z^2}{4} - \cancel{4}z = \frac{x}{4} - 5$

$(y-2)^2 + \left(\frac{z}{2} - 1\right)^2 = \frac{x}{4} - 5 + 5$

$(y-2)^2 + \left(\frac{z}{2} - 1\right)^2 = \frac{x}{4}$ elliptic paraboloid

<u>To know</u>

- graphs of ellipses, circles, hyperbola, parabola
- sketch using "traces"