Lecture 3

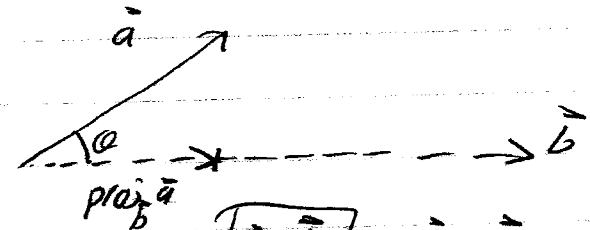
Review $\vec{a} = (a_1, a_2, a_3, \ldots, a_n)$ $\vec{b} = (b_1, b_2, b_3, \ldots, b_n)$

dot product $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$

Key Fact $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$ where θ is angle between $\vec{a}$ & $\vec{b}$.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|a||b|}$$

Projections Given $\vec{a}, \vec{b}$



1. $\text{comp}_{\vec{a}} \vec{b} = \text{scalar proj} = |\vec{a}| \cos\theta = \boxed{\frac{\vec{a} \cdot \vec{b}}{|a|}} = \vec{b} \cdot \frac{\vec{a}}{|a|}$

   Rmk $\frac{\vec{a}}{|a|}$ is unit vector in same direction

2. $\text{proj}_{\vec{a}} \vec{b} = \text{comp}_{\vec{a}} b \cdot \frac{\vec{a}}{|a|} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \cdot \vec{a}$

$\boxed{\text{Work} = \vec{F} \cdot \vec{D}}$

CROSSPRODUCT

Def Let $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Remarks

1. Input is 2 three-dimensional vectors.
   Output is a 3 dimension vector.
   $$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

2. $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{3x3 determinant}$$
   $$= \vec{i}(a_2 b_3 - a_3 b_2) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

3. $\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$
   Thus $\boxed{\vec{a} \times \vec{a} = \vec{0}}$

Ex $(2,1,3) \times (-3,1,5) = (5-3, -9-10, 2--3) = (2,-19,5)$

Key Thm

1. $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$
   (i.e. $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0 = (\vec{a} \times \vec{b}) \cdot \vec{b}$

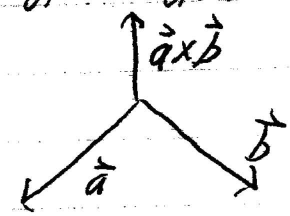
2. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$
   COR $\vec{a}, \vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$

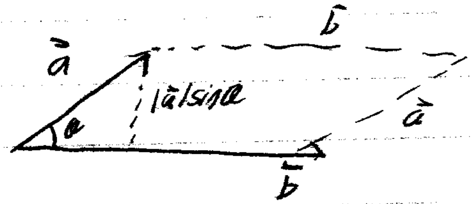
Proof

1. Just do it, everything cancels!
2. Do it, use $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$ plus $\cos^2 + \sin^2 = 1$

We know $|\vec{a} \times \vec{b}|$ and almost which way it points.

Right-Hand Rule: Curl right fingers from $\vec{a}$ to $\vec{b}$ thumb points in direction of $\vec{a}\times\vec{b}$:

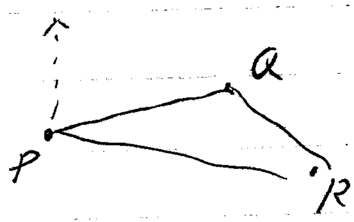




COR $|\vec{a}\times\vec{b}|$ = area of parallelogram formed by $\vec{a}$ & $\vec{b}$.

Problem

1. Find a vector perpendicular to plane containing $P=(2,-1,1)$, $Q=(1,1,1)$ and $(0,-3,2)=R$



$\vec{PQ} = (-1,2,0)$
$\vec{PR} = (-2,2,1)$

$\vec{PQ}\times\vec{PR} = \boxed{(2,1,2)}$

2. Find area of $\triangle PQR$

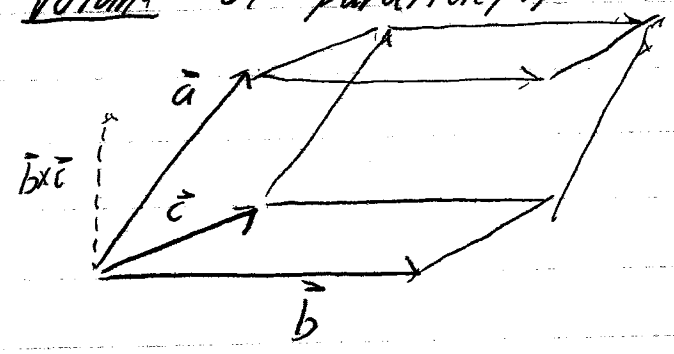
   , It is ½ area of parallelogram

$$= \frac{1}{2}|\vec{PQ}\times\vec{PR}|$$

$$= \frac{1}{2}\sqrt{4+1+4} = \boxed{3/2}$$

Scalar Triple Product

Problem Find volume of parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$.



$V = (\text{area base})\cdot \text{height}$

1. area base $= |\vec{b}\times\vec{c}|$

2. Height = component of $\vec{a}$ in direction $\vec{b}\times\vec{c}$

$$= \frac{\vec{a}\cdot\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$$

Conclude

$|\vec{a}\cdot\vec{b}\times\vec{c}|$ = Volume of parallelipided

absolute value

Def $\vec{a}\cdot(\vec{b}\times\vec{c})$ called scalar tiple product.

It is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Ex
Show that vectors $\underset{\vec{a}}{(1,5,-2)}$, $\underset{\vec{b}}{(3,-1,0)}$ and $\underset{\vec{c}}{(5,9,-4)}$ are coplanar.

Ans All in same plane $\longleftrightarrow$ Vol of p.p. is 0.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (1,5,-2)\cdot(-4, 12, 32) = -4-60+64 = 0.$$

Ex Find volume of parallelepided w/ adjacent edges PQ, PR, PS

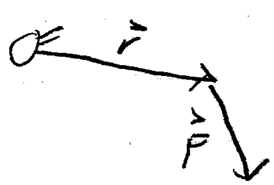
$P=(2,0,-1)$ $Q=(4,1,0)$, $R=(3,-1,1)$ $S=(2,-2,2)$

$\vec{PQ}=(2,1,1)$ $\vec{PR}=(1,-1,2)$ $\vec{PS}=(0,-2,3)$

$$\vec{PQ}\cdot(\vec{PR}\times\vec{PS}) = (2,1,1)\cdot(1,3,-2) = (2+3-2) = \boxed{3}$$

Ex Does $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ make sense? Is it scalar or vector?
$\vec{a}\times(\vec{b}\cdot\vec{c})$ " " "
$\vec{a}\times(\vec{b}\times\vec{c})$

Physics Fact Force $\vec{F}$ acting on rigid body at point given by pos vector $\vec{r}$.

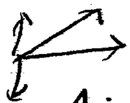


Torque $= \tau = \boxed{\vec{r}\times\vec{F}}$

measures tendency to rotate about origin.

Ex

Find <u>two</u> unit vectors $\perp$ to $(1,2,-5)$ and $(1,1,-1)$



A: $\vec{a}\times\vec{b} = (3,-4,-1)$

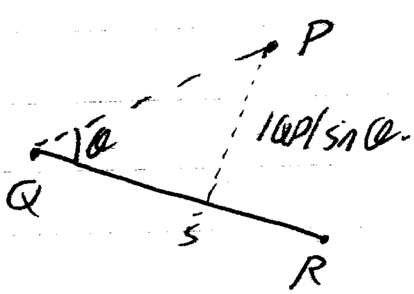
$|\vec{a}\times\vec{b}| = \sqrt{9+16+1} = \sqrt{26}$

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \boxed{\begin{array}{l}\frac{1}{\sqrt{26}}(3,-4,-1) \\ \text{OR} \\ -\frac{1}{\sqrt{26}}(3,-4,-1)\end{array}}$$

<u>Ex</u> Let P be a point on line L passing through Q & R. Show distance from P to L is

$$\frac{|\overrightarrow{QR}\times\overrightarrow{QP}|}{|\overrightarrow{QR}|} = \frac{|QR|\,|QP|\sin\theta}{|QR|} = |QP|\sin\theta$$

<u>Proof</u>



Ex Prove Prop 6 of Thm 8

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$