Lecture 23

Review Curves $\mathcal{C}: \vec{r}(t)\quad a \le t \le b \qquad ds = |\vec{r}\,'(t)|\,dt$

function $f(x,y)$, $\quad \int_{\mathcal{C}} f(x,y)\,ds = \boxed{\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt} \; *$

Special case: $\vec{F}(x,y) = (P,Q)$ vector field, $\vec{T}$ unit tangent,

integrate $\vec{F}\cdot\vec{T}$, add up amount of force tang to curve:

$$\text{Work} = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \boxed{\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r}} \; \star$$

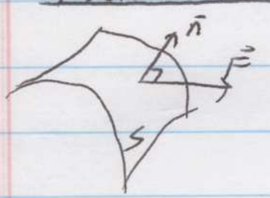
Last Time Surface S given by $\vec{r}(u,v): \underset{\mathbb{R}^2}{D} \longrightarrow \mathbb{R}^3$

function $f(x,y,z)$

$dS = |\vec{r}_u \times \vec{r}_v|\,dA$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA} \; *$$

Special case: $\vec{F}(x,y,z) = (P,Q,R)$ vector field



Want component of $\vec{F} \perp$ to S

$\vec{n}$ = unit normal. We want

$$\iint_S \vec{F}\cdot\vec{n}\,dS$$

Problem Is the surface orientable? For curve, "orientation" given by increasing t.

Define Orientable, upward pointing normal, pos orientation for closed surface.

For $z = f(x,y)$ $\quad \vec{r}(u,v) = (u, v, f(u,v))$

$$r_u = (1, 0, \tfrac{\partial f}{\partial u}) \quad r_v = (0, 1, \tfrac{\partial f}{\partial v})$$

$$r_u \times r_v = (-\tfrac{\partial f}{\partial u}, -\tfrac{\partial f}{\partial v}, 1) \text{ pts up}$$

In general $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ so:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint \vec{F}(r(u,v)) \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \cdot |r_u \times r_v|\, dA$$

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA} \star$$

called Flux of $\vec{F}$ across S

Ex Find the flux of $\vec{F}(x,y,z) = (xy, yz, zx)$ across the paraboloid $z = 4 - x^2 - y^2$ above $[0,1] \times [0,1]$ w/ upward orientation.

A: $\vec{r}(u,v) = (u, v, 4 - u^2 - v^2) \quad 0 \le u \le 1 \quad 0 \le v \le 1$

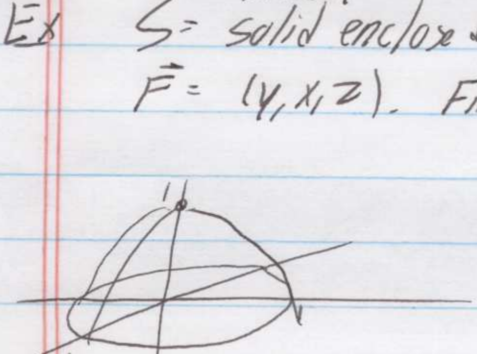
$\vec{r}_u = (1, 0, -2u) \quad \vec{r}_v = (0, 1, -2v) \qquad r_u \times r_v = (2u, 2v, 1)$

$$\int_0^1 \int_0^1 (uv, v(4-u^2-v^2), u(4-u^2-v^2)) \cdot (2u, 2v, 1)\, du\, dv$$

$$= \int_0^1 \int_0^1 2u^2 v + 2v^2(4-u^2-v^2) + u(4-u^2-v^2)\, du\, dv = \_\,\_\,\_$$

Ex S = boundary of solid enclosed by $z = 1 - x^2 - y^2$ & $z = 0$

$\vec{F} = (y, x, z)$. Find $\iint_S \vec{F} \cdot \vec{n}\, dS$

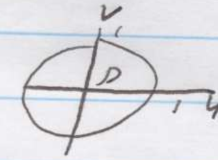


Let S_1 = curved part

S_2 = Disc $x^2 + y^2 \le 1$

S_1: $\vec{r}(u,v) = (u, v, 1-u^2-v^2)$ $\quad r_u \times r_v = (2u, 2v, 1)$

$u, v \in$ circle radius 1



$$\iint_{S_1} (v, u, 1-u^2-v^2)\cdot(2u, 2v, 1)\,du\,dv$$

$$= \iint 4uv + 1 - u^2 - v^2 \,du\,dv \qquad \text{Let } u = r\cos\theta \;\; v = r\sin\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r^2\sin\theta\cos\theta + 1 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r^3\sin\theta\cos\theta + r - r^3 \,dr\,d\theta$$

$$= \int_0^{2\pi} r^4\sin\theta\cos\theta + \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1$$

$$= \int_0^{2\pi} \sin\theta\cos\theta + 1/4\,d\theta = \frac{1}{2}\sin^2\theta + \frac{1}{4}\theta\Big|_0^{2\pi}$$

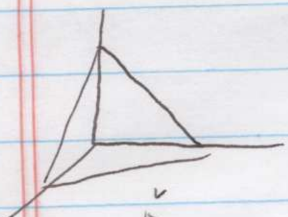
$$= \pi/2$$

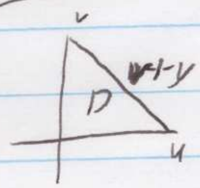
S_2: $\vec{n} = (0,0,1)$

$$\iint_{S_2} \vec{F}\cdot\vec{n} = \iint_{S_2} -z\,dA = 0$$ (or do $\vec{r}(u,v) = (u,v,0)$
$r_u \times r_v = (1,0,0)\times(0,1,0)$
$= (0,0,1)$

Final answer $\pi/2$

Ex Find flux of $\vec{F} = (xze^y, -xze^y, z)$ across plane $x+y+z=1$ in 1^{st} octant.





parametrize $r(u,v) = (u, v, 1-u-v)$ $(u,v) \in D$

$r_u = (1,0,-1)$ $r_v = (0,1,-1)$

$r_u \times r_v = (1,1,1)$ idiot!

$\vec{n} = (-1,-1,-1)$

$$-\iint_D \vec{F}\cdot\vec{n} = -\iint_D (1-u-v)\,du\,dv = -\int_0^1 \int_0^{1-u} 1-u-v\,dv\,du$$

$$= -\int_0^1 \left(v - uv - v^2/2\right)\Big|_0^{1-u}$$

$$= -\int_0^1 1-u-u(1-u) - \frac{(1-u)^2}{2} = -1/6$$

Calculating Flux

1. Parametrize surface $\vec{r}(u,v)$ $(u,v) \in D$
2. Get $\vec{n} = \vec{r}_u \times \vec{r}_v$, check direction, use $-\vec{n}$ if needed
3. Compute $\iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{n}\,dA$

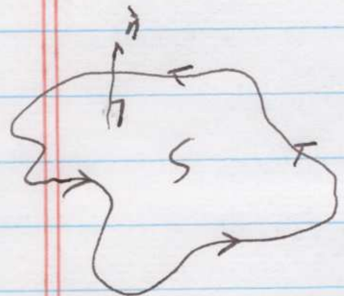
Recall

Green's Thm $\oint_{C} \vec{F}\cdot d\vec{r} = \iint_{S} \underbrace{\text{curl}\,\vec{F}\cdot k}_{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}\, dA$

Stokes' Thm

S an oriented, piecewise smooth surface, bounded by simple, closed, piecewise smooth boundary curve C w/ pos orientation.
$\vec{F}$ a vector field w/ continuous partials on open set containing S.
Then

$$\oint_{C} \vec{F}\cdot d\vec{r} = \iint_{C} \text{curl}\,\vec{F}\cdot d\vec{S}$$



$$= \iint_{D} \text{cur}\,\vec{F}\cdot\vec{n}\, dS$$

can write $\iint_{S} \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

Like FTOC

Remark ~~then~~ If S lies in xy plane then $\vec{n}=\vec{k}$ and we get Green's Thm

Ex $\oint_{C} \vec{F}\cdot d\vec{r}$ $\quad \vec{F} = (-y^2, x, z^2)$ $\quad C$ is curve $\cap$ $y+z=2$, $x^2+y^2=1$

cclock from above