Review

D = uv plane, \( \mathbf{F}(u,v) = (x(u,v), y(u,v), z(u,v)) \) parametrized surface.

Examples

1. Graph of \( z = f(x,y) \) \( \mathbf{F}(u,v) = (u,v,f(u,v)) \)

2. Plane through point \((x_0,y_0,z_0)\) containing nonparallel vectors \( \mathbf{a} = (a_1, a_2, a_3) \quad \mathbf{b} = (b_1, b_2, b_3) \)

\[ \mathbf{r}(s,t) = (x_0,y_0,z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3) \]
\[ = (x_0 + sa_1 + tb_1, y_0 + sa_2 + tb_2, z_0 + sa_3 + tb_3) \]

3. Rotate curve \( y = f(x) \ a \leq x \leq b \) around x axis, \( \Theta = \text{constant} \)

\[ x = x \quad y = f(x) \cos \Theta \quad z = f(x) \sin \Theta \]

\[ \mathbf{r}(x, \Theta) = (x, f(x) \cos \Theta, f(x) \sin \Theta) \]

Tangent Planes

Review last class \( \mathbf{F}(u,v) = (x(u,v), y(u,v), z(u,v)) \)

\( \mathbf{r}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \) at a point \((u_0,v_0)\) is tangent vector to grid curve fixing \( v = v_0 \).

Similarly

\( \mathbf{r}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \) tangent to curve fixing \( u = u_0 \).
Fact: The tangent plane at point \( \tilde{f}(u_0, v_0) \) contains tangent vectors \( \tilde{r}_u(u_0, v_0) \) and \( \tilde{r}_v(u_0, v_0) \), so has \( \tilde{r}_u \times \tilde{r}_v \) as normal vector.

**Examples**

1. \( \tilde{r}(u, v) = (u, v, f(u, v)) \)
   \[ r_u = (1, 0, \frac{\partial f}{\partial u}) \quad r_v = (0, 1, \frac{\partial f}{\partial v}) \]
   \[ r_u \times r_v = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1) \]
   Already did this in 14. 41

2. Find tangent plane to \( \tilde{r}(u, v) = (u^2 + v, uv, u^2 - v^2) \) at point \( (2, 1, 0) \).

3. Find parametric equation for the part of the part of the elliptic paraboloid \( x + y^2 + z^2 = 4 \) that lies in front of the plane \( x = 0 \). Then find the tangent plane at \( (1, 1, 1) \).
Surface Area

Review \( \vec{R}(t) = a \cos b \)
To get arc length, we split curve into segments, each of length \( |\vec{R}(\alpha) - \vec{R}(\beta)| \), and

\[
    A_{arc} = \lim_{|\Delta t| \to 0} \frac{1}{|\Delta t|} |\vec{R}(\alpha) - \vec{R}(\beta)|
    = \int_{\alpha}^{\beta} \frac{|\vec{R}'(t)|}{|\Delta t|} \, dt
\]

For a surface, the corr. problem is surface area

Estimated area = area of parallelogram in any plane

\[ = |\vec{u} \times \vec{v}| \]

Def: Thin surface \( \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \) is a smooth parametric surface covered once as \( (u,v) \in D \).
Then the surface area is:

\[
    A(s) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA.
\]

Ex. Sphere radius \( a \)
\[
    (a \sin \theta \cos \phi, a \sin \theta \sin \phi, a \cos \phi)
\]
\[ 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi \]

Check \( \vec{r}_u \times \vec{r}_v = (a^2 \sin^2 \theta \cos \phi, a^2 \sin^2 \theta \sin \phi, a^2 \sin \theta \cos \phi) \)

Ex. Graph of a function \( \vec{r}(x,y) = (x,y, f(x,y)) \)

\[
    A = \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2} \, dA \\
    \text{compare to } A_{arc} = \int_{\alpha}^{\beta} \frac{|\vec{R}'(t)|}{|\Delta t|} \, dt
\]
Ex. Find s.A. of portion of saddle \( Z = x^2 - y^2 \) inside cylinder \( x^2 + y^2 = 1 \).

Parametric \( (x, y, x^2 - y^2) \); \( 0 < x < 1 \)

\[ R_x = (1, 0, 2x) \quad R_y = (0, 1, -2y) \]

\[ R_x \times R_y = (-2x, 2y, 1) \quad |R_x \times R_y| = \sqrt{1 + 4y^2 + 4y^2} \]

\[ \oint \frac{1}{\sqrt{1 + 4y^2 + 4y^2}} \, dV = \frac{1}{2} \oint \frac{1}{\sqrt{1 + y^2}} \, dx \, dy \]

Ex. Find area of \( \hat{F}(u, v) = (u \cos v, u \sin v, v) \) \( 0 < u < 1 \), \( 0 < v < \pi \)

Spiral ramp

Review: Line integral of vector field over curve.

At each point we add up \( \hat{F} \cdot \hat{t} \)

\[ \oint \hat{F} \cdot d\hat{s} = \int_a^b \hat{F}(u) \cdot \hat{t}(u) \, du \]

Integral of a function

\[ \oint \hat{F} \cdot d\hat{s} = \oint \hat{F} \cdot d\hat{t} \quad \text{order} \]

\[ \oint \hat{F} \cdot d\hat{s} = \oint \hat{F} \cdot d\hat{n} \quad \text{differential} \]
Surface Integrals

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \Delta S_{ij} \]

area \( A \) \( \|ru \times rv\| \Delta u \Delta v \)

**Def.** The surface integral of a function \( f(x, y, z) \) over a surface parametrized by \( \boldsymbol{r}(uv) : D \rightarrow \mathbb{R}^3 \) is

\[ \iint_D f(\boldsymbol{r}(uv)) \|ru \times rv\| \, dA \]

Compare to

\[ \frac{\partial F}{\partial t} \frac{\partial F}{\partial u} \Delta t \]

**Ex.** \[ \oint_C xy \, ds \quad \text{is plane } x+y+z=1 \text{ in 1st octant.} \]