Lecture 21

Review

1. If $\vec{F} = (P(x), Q(x))$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on open, simply conn. region then $\vec{F}$ is conservative. Today we generalize this to 3-dim fields.

   - Partial integration helps us find a potential function f which we can use to do line integrals using $\int_C \nabla f\, dr = f(r(b)) - f(r(a))$

2. Green's Thm C closed curve, simple, pos oriented w/ D inside. Then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- When $\vec{F}$ is conservative this confirms what we know, that line integral along closed curve is zero.

- Area $D = \int x\,dy = \int y\,dx = \frac{1}{2} \int x\,dy + y\,dx$

Ex $\vec{F} = \langle e^x + x^2 y,\ e^y - xy^2 \rangle$ C = circle $x^2 + y^2 = 9$ clockwise. Find

$$\int_C \vec{F} \cdot d\vec{r}$$

Vector Differential Operators

Notation $\nabla =$ "del" $= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ or $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ etc.

think of it as an operator.

Ex $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ gradient.

Def Let $\vec{F}$ be a vector field on $\mathbb{R}^3$, $\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$. Then curl $\vec{F}$ is a new vector field:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Ex $\vec{F} = (3x^2, xy, z)$ $\quad$ curl $\vec{F} = (0, 0, y)$

Thm 1. Suppose $f(x,y,z)$ has continuous 2^{nd} partials. Then

$$\text{curl}(\nabla f) = 0.$$

Thus conservative vector fields have curl 0

2. Suppose $\vec{F}$ is continuous on all of $\mathbb{R}^3$. Then if curl $\vec{F} = 0$, $\vec{F}$ is conservative

Def If curl $\vec{F} = 0$ at a point, say $\vec{F}$ is irrotational at P.

Ex

rotation in plane $\perp$ to curl.

Special case $\quad \vec{F}(x,y) = (P(x,y), Q(x,y), R(x,y))$

$$\text{curl}\,\vec{F} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Def: Suppose $\vec{F} = (P, Q, R)$ a vector field. Then the divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- curl $\vec{F}$ is a vector field, div $\vec{F}$ is a scalar.

Exercise: $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

Proof: Clairaut, cancel in pairs

Rmk: div $\vec{F}$ measures tendency of fluid to flow from a point.
If $\operatorname{div} \vec{F} = 0$, $\vec{F}$ is incompressible.

Restating Green Thm in Vector Form

1. Think $\vec{F} = (P, Q, 0)$ so $\operatorname{curl} \vec{F} = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \vec{k} \, dA$$

2. Recall $\vec{n}(t)$ unit normal to curve $= \left(\frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|} \right)$ plug in

$$\int_C \vec{F} \cdot \vec{n} \, ds = \iint_D \operatorname{div} \vec{F}(x, y)$$

We generalize these at end of course.

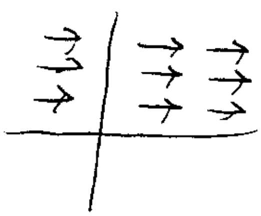
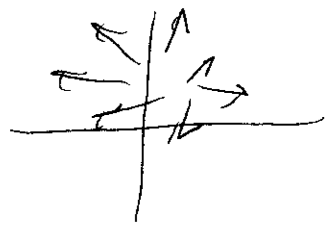
Problems

1. $\vec{F}(x,y,z) = (xyz^2, x\sin y, x+yz)$ Find curl $\vec{F}$, div $\vec{F}$.

2. Do the following make sense? f a function, $\vec{F}$ a vector field

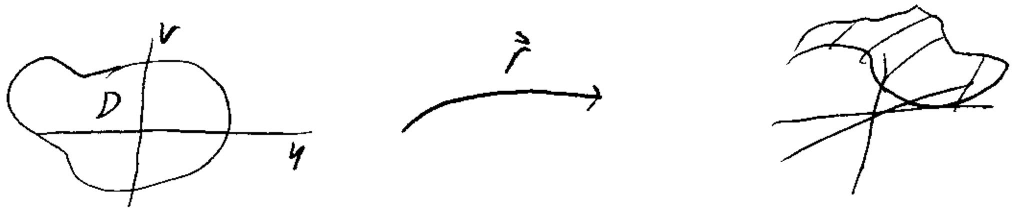
   curl f
   curl grad A
   div(curl(grad f))

3. Is there a vector field $\vec{G}$ such that curl $\vec{G} = \langle x\sin y, \cos y, z-xy \rangle$?

4.   discuss curl $\vec{F}$, div $\vec{F}$

Parametric Surfaces

Ex $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ $(u,v) \in D \subseteq \mathbb{R}^2$



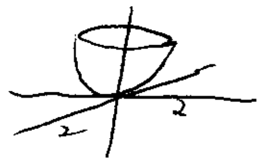
Ex 1. $\vec{r}(u,v) = (u, v, f(u,v))$ is graph of function

2. $\vec{r}(u,v) = (\sin v\cos u, \sin v\sin u, \cos v)$ $0 \le u \le 2\pi$, $0 \le v \le \pi$

Sphere radius 1

3. $\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, r)$ $\quad 0 \le r \le 2 \quad 0 \le \theta \le 2\pi$

Notice $z = x^2 + y^2$



piece of paraboloid

Parametrizing surfaces is difficult!

<u>Next class</u>

- tangent planes (analogous to tangent vector)
- surface areas (" " arc length)
- surface integrals (" " line integrals)