1. If \( \vec{F} = (P(x,y), Q(x,y)) \) and \( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \) on open, simply connected region then \( \vec{F} \) is conservative. Today we generalize this to 3-dim fields.

- Partial integration helps us find a potential function \( f \) which we can use to do line integrals using
  \[
  \oint_{\Gamma} \vec{F} \cdot d\vec{r} = f(\Gamma(\text{end})) - f(\Gamma(\text{start}))
  \]

2. **Green's Theorem** \( C \) closed curve, simple, positively oriented with \( D \) inside. Then

\[
\oint_{C} P \, dx + Q \, dy = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

- When \( \vec{F} \) is conservative this confirms what we know that line integral along closed curve is zero.

- Area \( D \): \( \int S = \iint_{D} \, dA = \frac{1}{2} \iint_{D} (x \, dy - y \, dx) \)

- **Ex:** \( \vec{F} = <e^y + x^2 y, e^x - xy^2> \) \( C \) = circle \( x^2 + y^2 = 9 \) clockwise. Find

\[
\oint_{C} \vec{F} \cdot d\vec{r}
\]
Vector Differential Operators

Notation \( \nabla = \triangledown = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \) or \( (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \) etc.,
think of it as an operator.

Ex \( \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \) gradient.

Def Let \( \vec{F} \) be a vector field on \( \mathbb{R}^3 \), \( \vec{F} = (P, Q, R) \). Then curl \( \vec{F} \) is a new vector field:

\[
\text{curl } \vec{F} = \nabla \times \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
\]

Ex \( \vec{F} = (3x^2, xy, z) \) curl \( \vec{F} = (0, 0, y) \)

Thm 1 Suppose \( f(x, y, z) \) has continuous 2nd partials. Then curl \( \nabla f = 0 \).
This conservative vector fields have curl 0

2. Suppose \( \vec{F} \) is continuous on all of \( \mathbb{R}^3 \) then if curl \( \vec{F} = 0 \), \( \vec{F} \) is conservative

Def If curl \( \vec{F} = 0 \), say \( \vec{F} \) is irrotational at \( P \).

Imp \( \nabla \times \vec{F} \)
rotation in plane \( L \) to \( \vec{F} \)

Special case \( \vec{F}(x, y) = (P(x, y), Q(x, y), R(x, y)) \)

\[
\text{curl } \vec{F} = \left( 0, 0, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial y} \right)
\]
**Def:** Suppose \( \mathbf{F} = (P,Q,R) \) a vector field. Then its divergence \( \mathbf{F} \) is

\[
\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

\( \text{curl } \mathbf{F} \) is a vector field, \( \text{div } \mathbf{F} \) is a scalar.

**Exercise** \( \text{div } (\text{curl } \mathbf{F}) = 0 \)

Because \( \text{c} \text{i} \text{a} \text{i} \text{r} \text{a} \text{n} \text{i} \text{t} \text{ u} \text{n} \text{e} \text{s} \) cancel in pairs

**Rmk:** \( \text{div } \mathbf{F} \) measures tendency of fluid to flow from a point.

\( \text{If } \text{div } \mathbf{F} = 0 \) \( \mathbf{F} \) is incompressible.

**Restating Green’s Theorem in Vector Form**

1. \( \text{Think } \mathbf{F} = (P,Q,0) \) so \( \text{curl } \mathbf{F} = (0,0, -\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}) \). Then

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl } \mathbf{F} \cdot \hat{n} \, dA
\]

2. \( \text{Recall } \hat{n} \) (unit normal to curve) = \( (\frac{1}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{1}{\sqrt{10}}) \) plug in

\[
\oint \mathbf{F} \cdot \hat{n} \, ds = \iint_D \text{div } \mathbf{F} \, dA
\]

We generalize these at end of course.
Problems

1. \( \vec{F}(x,y,z) = (xyz^2, x \sin y, xz) \) Find \( \text{curl} \vec{F}, \text{div} \vec{F} \).

2. Do the following make sense? Let \( \vec{F} \) a vector field:
   \[ \text{curl} \vec{F} \]
   \[ \text{curl}(\text{grad} A) \]
   \[ \text{div}(\text{curl}(\text{grad} f)) \]

3. Is there a vector field \( \vec{G} \) such that \( \text{curl} \vec{G} = \langle x \sin y, \cos y, z - y \rangle \)?

4. \[ \begin{array}{c}
   \vec{F} \\
   \vec{G}
   \end{array} \]
   Discuss \( \text{curl} \vec{F} \), \( \text{div} \vec{F} \).

---

Parametric Surfaces

Ex. \( \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \) \( (u,v) \in D \subseteq \mathbb{R}^2 \)

Ex.

1. \( \vec{r}(u,v) = (u, v, \sin(u \sin v)) \) is graph of function.

2. \( \vec{r}(u,v) = (\sin v \cos u, \sin v \sin u, \cos v) \) \( 0 \leq u \leq 2\pi \)
   \[ 0 \leq v \leq \pi \]

   Sphere radius 1
\[ \hat{r}(r, \theta) = (r \cos \theta, r \sin \theta, r) \quad 0 \leq r \leq 2 \quad 0 \leq \theta \leq 2\pi \]

Notice: \[ z = x^2 + y^2 \]

piece of paraboloid

Parametrizing surfaces is difficult!

Next class

- tangent planes (analogues to tangent vectors)
- surface areas ("""" are lengths)
- surface integrals ("""" line integrals)