Recall \( F(0)dx = F(b) - F(a) \)

Then let \( \vec{r}(t) \) be a smooth curve. Suppose \( f \) is a function with \( \nabla f \) continuous on the curve. Then

\[
\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))
\]

This theorem says:

<table>
<thead>
<tr>
<th>Line integral of conservative vector field</th>
<th>=</th>
<th>change in potential</th>
</tr>
</thead>
</table>

Ex. \( \vec{F}(x,y) = (y,x) \). Find work done along \( \vec{r}(t) = (t, \sqrt{2t}) \) from \( (1,1) \) to \( (3,3) \).

Notice \( f(x,y) = xy \) \( \vec{F} = \nabla f \). Thus

\[
\int_1^3 \vec{F} \cdot d\vec{r} = f(3) - f(1,1) = 27 - 1 = 26
\]

Check this is

\[
\int_1^3 \left( \sqrt{2t} \frac{1}{2\sqrt{t}} \right) dt = \int_1^3 \sqrt{2} dt = \sqrt{2} \left[ t \right]_1^3 = 2\sqrt{2}
\]

Def. \( \int_1^3 \vec{F} \cdot d\vec{r} \) is independent of path in Dir

\[
\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_0} \vec{F} \cdot d\vec{r} \text{ for any curves } C_1, C_0 \text{ in } D \text{ with same start and endpoint.}
\]

Ex. Line integrals of conservative vector fields are ind of path.
Then $\oint \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$ if and only if

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for all closed curves } C \text{ in } D$$

**Proof**

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint \mathbf{F} \cdot d\mathbf{r}$$

$$= \oint \mathbf{F} \cdot d\mathbf{r} - \oint \mathbf{F} \cdot d\mathbf{r}$$

Then $\oint \mathbf{F} \cdot d\mathbf{r}$ is ind of path if and only if $\mathbf{F}$ is conservative in $D$.

**Proof**

Define $f(x,y) = \oint \mathbf{F} \cdot d\mathbf{r}$ where $(a,b)$ fixed.

Choose any path to $(x,y)$. Check $\nabla f = \nabla \mathbf{F}$.

Open: Any pt in $D$ has

Connected: Path b/w any two points

**Problem**: When is $\mathbf{F}$ conservative? Suppose $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$.

If $P = \frac{\partial f}{\partial y}$ and $Q = \frac{\partial f}{\partial x}$ then Clairaut says $P_y = Q_x$.!
Let $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ be conservative, with $P, Q$ having continuous partial derivatives. Then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

2. Suppose $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout $D$. Suppose $D$ is open, simply connected. Then $\mathbf{F}$ is conservative.

Simply connected: simple closed curves can be pulled shut.

Ex. $\mathbf{F}(x,y) = (x^3, y^2) \quad \frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$ so $\mathbf{F}$ is conservative.

$$f = \frac{1}{3} x^3 + \frac{1}{2} y^2 \quad \nabla f = \mathbf{F}$$

Ex. $\mathbf{F}(x,y) = (-y, x) \quad \frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$ Not conservative.

Ex. Let $\mathbf{F} = (ye^{x+y} \sin y, e^x \cos y)$

Is $\mathbf{F}$ conservative? If so, find $f$. 
Ex. Show \( \int_C (x \, dx + y \, dy) \) is independent of path and evaluate the integral \( C \) goes from \((1,0)\) to \((2,0)\).

Green's Thm

Define a simple, closed curve \( C \), positive orientation means once around \( C \).

Then let \( C \) be positively oriented, piecewise-smooth, simple closed curve in the plane. Let \( D \) be the region bounded by \( C \). If \( P \) and \( Q \) have continuous partial derivatives on an open region containing \( D \) then:

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

Proofs:
1. LHS = \( \iint_D \vec{F} \cdot d\vec{S} \), \( \vec{F} = \{P,Q\} \)
2. Also written:

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_C P \, dx + Q \, dy
\]
**Example 1:**
\[ \int \cos(x^3) \, dx + xy \, dy \]  
**C triangular curve**  
\[(0,0) \rightarrow (1,0) \rightarrow (1,3)\]

**Example 2:**
\[ \int (ye^{y^2}) \, dx + (2x + \cos(y^2)) \, dy \]  
**C boundary**  
of region enclosed by \( y = x^2, x = y^2 \)

**Example 3:**
\[ F(x,y) = (0, x) \]  
\[ \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} = 1 \]

**Example 4:**
\[ F(x,y) = (1-y, 0) \]
\[ F(x,y) = (y, x/y) \]

Thus, \[ \int \int_D dA = \text{Area D} = \oint x \, dy = -\oint y \, dx \]

**Example 5:**
**Area of ellipse**  
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]  
using \( 3 \cdot 1 \)