Lecture 20

Recall FTOC. $\int_a^b F'(x)\,dx = F(b) - F(a)$

Thm Let $\vec{r}(t)$ $a \le t \le b$ be a smooth curve. Suppose f is a function w/ ∇f continuous on the curve. Then

$$\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Rmk This theorem says:

Line integral of conservative vector field = change in potential.

Ex $\vec{F}(x,y) = (y, x)$. Find work done along $\vec{r}(t) = (t, \sqrt{t})$ from $(1,1)$ to $(9,3)$.

Notice $f(x,y) = xy$ $\vec{F} = \nabla f$. Thus

$$\int_1^9 \vec{F} \cdot d\vec{r} = f(9,3) - f(1,1) = 27 - 1 = 26$$

Check this is $\int_1^9 (\sqrt{t}, t) \cdot (1, \frac{1}{2\sqrt{t}})\,dt$

$$= \int_1^9 \sqrt{t} + \tfrac{1}{2}\sqrt{t}\,dt = \int_1^9 \tfrac{3}{2} t^{1/2}\,dt = t^{3/2}\Big|_1^9 = 26$$

Def $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> in D if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \quad \text{any curves } C_1, C_2 \subseteq D$$

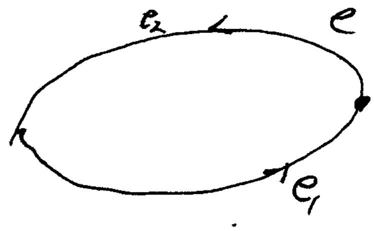
w/ same start & endpoint.

Ex Line integrals of conservative vector fields are ind. of path.

<u>Thm</u> $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if & only if

$$\int_C \vec{F}\cdot d\vec{r} = 0 \quad \text{for all } \underline{\text{closed}} \text{ curves } C \text{ in } D$$

<u>Proof</u>



$$\int_C = \int_{C_1} + \int_{C_2} = \int_{C_1} - \int_{-C_2} = 0$$

<u>Thm</u> Suppose $\vec{F}$ is a continuous vector field on an <u>open</u> <u>connected</u> region D. Then

$\int_C \vec{F}\cdot d\vec{r}$ is ind. of path in D if & only if $\vec{F}$ is conservative.

<u>Proof</u> $\leftarrow$ done

$\rightarrow$ Define $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$ where (a,b) fixed,

choose <u>any</u> path to (x,y). Check $\nabla f = \vec{F}$. //

<u>Open</u>: Any pt in D has

<u>Connected</u> Path btw any two points

<u>Problem</u>: When is $\vec{F}$ conservative? Suppose $\vec{F}(x,y) = (P(x,y), Q(x,y))$!

If $P = \frac{\partial f}{\partial x}$ & $Q = \frac{\partial f}{\partial y}$ then Clairaut says

$P_y = Q_x$!

Thm

1. Let $\vec{F}(x,y) = (P(x,y), Q(x,y))$ be conservative, where P, Q have continuous partial derivatives. Then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

2. Suppose $\vec{F}(x,y) = (P(x,y), Q(x,y))$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Suppose D is open, simply connected. Then $\vec{F}$ is conservative

simply connected: simple closed curves can be "pulled shut"

Ex $\vec{F}(x,y) = (x^2, y^2)$ on $\mathbb{R}^2$ $\qquad \frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$ so F is conservative.

$$f = \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 \qquad \nabla f = F$$

Ex $F(x,y) = (-y, x)$ $\qquad \frac{\partial P}{\partial y} = -1, \ \frac{\partial Q}{\partial x} = 1$ Not conservative

Ex Let $\vec{F} = (ye^x + \sin y, e^x + x\cos y)$

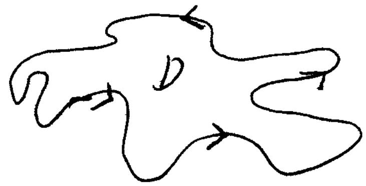
Is $\vec{F}$ conservative? If so, find f.

Ex. Show $\int_C \tan y\, dx + x\sec^2 y\, dy$ is ind of path and evaluate the integral, C goes from $(1,0)$ to $(2, \pi/4)$

Ex.

Green's Thm

Def Given a simple, closed curve C, positive orientation means once around c.cwise (so region D on left).



Thm Let C be positively oriented, piecewise-smooth, simple closed curve in the plane. Let D be the region bounded by C. If P & Q have continuous partial derivatives on an open region containing D then:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Rmks 1. LHS $= \int_C \vec{F}\cdot d\vec{r}$ $\quad F = (P, Q)$

2. Also written $\oint_C$ or

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Ex $\oint_C \cos(x^3)\,dx + xy\,dy$ C triangular curve $(0,0) \to (2,0) \to (0,3)$

Ex $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos(y^2))\,dy$ C boundary

of region enclosed by $y = x^2,\ x = y^2$

Ex $\vec{F}(x,y) = (0, x)$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$\vec{F}(x,y) = (-y, 0)$ ''

$\vec{F}(x,y) = (-\tfrac{1}{2}y, \tfrac{1}{2}x)$ ''

Thus $\iint_D 1\,dA = \text{Area } D = \oint_C x\,dy = -\oint_C y\,dx$

$= \frac{1}{2}\oint_C x\,dy - y\,dx$

Ex Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using 3rd.