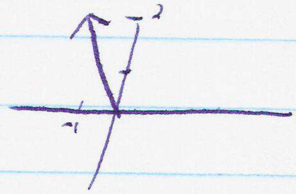
Math 241 Lecture 2

Review

- by using coordinates we can treat vectors algebraically.

Ex $\vec{v} = (-1, 2)$

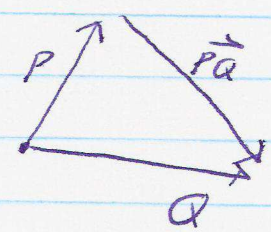


Vector operations of addition, scalar mult agree with our first description in terms of arrows.

Facts

1. If $P = (x_1, y_1, z_1)$ $Q = (x_2, y_2, z_2)$ then

$\overrightarrow{PQ} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$



2. $\vec{v} = (x, y)$ $\quad |\vec{v}| = \sqrt{x^2 + y^2}$ $\quad$ <u>length</u> or <u>magnitude</u> of $\vec{v}$

$\vec{v} = (x, y, z)$ $\quad |\vec{v}| = \sqrt{x^2 + y^2 + z^2}$ $\quad$ etc..

Notice $|\vec{v}|$ = Distance from (x, y, z) to $(0,0,0)$, as expected.

3. Usual dist. laws, etc.. hold, see p. 774

$$(3+5)\vec{v} = 8\vec{v} = 3\vec{v} + 5\vec{v}$$
$$3(\vec{u} + \vec{v}) = 3\vec{u} + 3\vec{v}$$
$$\vec{u} + \vec{0} = \vec{u}$$

etc..

Special vectors in $\mathbb{R}^3$

$\vec{i} = (1,0,0) \quad \vec{j} = (0,1,0) \quad \vec{k} = (0,0,1)$

Ex 1. $(2,-3,1) = 2\vec{i} - 3\vec{j} + \vec{k}$

2. $\vec{a} = \vec{i} + \vec{j} - \vec{k} \quad \vec{b} = 2\vec{i} - 3\vec{j}$ Then $\vec{a} + \vec{b} = (3\vec{i} - 2\vec{j} - \vec{k})$

Def Unit vector is a vector of length 1, Given $\vec{v} \neq 0$, $\frac{\vec{v}}{|\vec{v}|}$ always unit

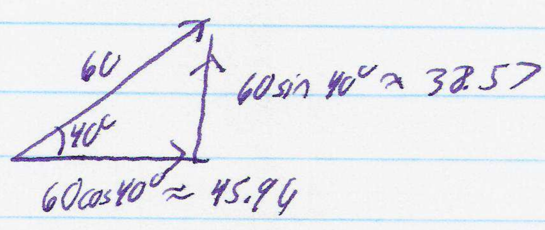
Ex Find a unit vector in the same direction as $\vec{v} = (1,2,-3)$

A: $|\vec{v}| = \sqrt{1+4+9} = \sqrt{14}$

$\vec{u} = \frac{1}{\sqrt{14}}(1,2,-3) = \left(1/\sqrt{14},\ 2/\sqrt{14},\ -3/\sqrt{14}\right)$

Physics Problems

#27 A quarterback throws football angle elevation $40°$ and speed 60 ft/sec. Find horizontal & vertical components of velocity vector.



#31 A woman walks west on the deck of a ship at 3 mph
Ship moves north at 22 mph
Find speed & direction of woman relative to water.

Dot Product

Def $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Input: 2 vectors Output: scalar

- Thus $\vec{a} \cdot \vec{b} \cdot \vec{c}$ makes no sense
- Defined for any dimension

$$(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Ex $(3, 1, -1) \cdot (1/2, 5, 2) = 3/2 + 5 - 2 = 9/2$

Basic Properties

1. $\vec{a} \cdot \vec{a} = |a|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b})$ $\lambda \in \mathbb{R}$
5. $\vec{0} \cdot \vec{a} = 0$

Different zeroes!

Proofs Check w/ formula using basic arithmetic rules.

<u>Why dot product ?</u>

<u>Theorem</u> Let θ be the angle between $\vec{a}$ + $\vec{b}$. Then

$$** \quad \boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta} \quad **$$

<u>Ex</u> Find angle between $(1,1,2,3) = \vec{v}$, $(-1,0,1,6) = \vec{u}$

$\vec{u}\cdot\vec{v} = 19$

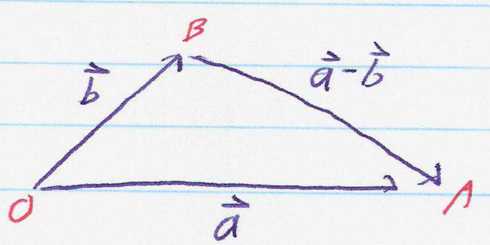
$|\vec{u}| = \sqrt{38}$ $\quad |\vec{v}| = \sqrt{15}$

$$\cos\theta = \frac{19}{\sqrt{38}\sqrt{15}} \approx .794$$

$$\boxed{\theta \sim 37.27^\circ}$$

This can be taken as <u>definition</u> of angle if you like.

<u>Proof</u>



Apply Law of cosines to ΔOAB

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA|\,|OB|\cos\theta$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|a||b|\cos\theta$$

expand & cancel!

<u>COROLLARY</u>

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular if & only if $\vec{a}\cdot\vec{b} = 0$

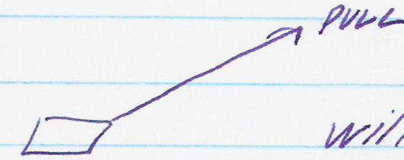
Def Given $\vec{a} = (a_1, a_2, a_3)$ the <u>direction angles</u> α, β, γ are $\angle$'s $\vec{a}$ makes w/ positive x, y, z axes. Thus:

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|a||i|} = \frac{a_1}{|a|} \qquad \cos\beta = \frac{a_2}{|a|} \qquad \cos\gamma = \frac{a_3}{|a|}$$

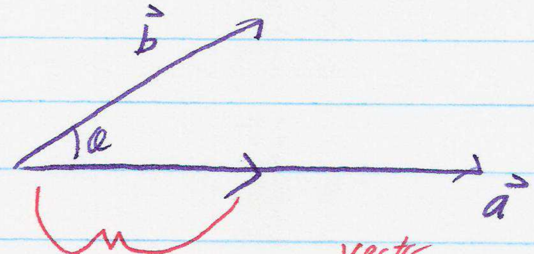
$$\vec{a} = |\vec{a}| \cdot (\cos\alpha, \cos\beta, \cos\gamma)$$

PROJECTIONS

Often we want the component of a vector in a certain direction.



will box slide?



$\text{proj}_{\vec{a}} \vec{b}$ vector projection of $\vec{b}$ onto a

$\text{comp}_a \vec{b}$ component of $\vec{b}$ along $\vec{a}$ $=$ length $|\text{proj}_{\vec{a}} \vec{b}|$

1. Compute $\text{comp}_a \vec{b}$ it is

$$\cos\theta = \frac{\text{comp}_a \vec{b}}{|\vec{b}|} \qquad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||b|} = \frac{\text{comp}_a \vec{b}}{|\vec{b}|}$$

$$\boxed{\text{comp}_a \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

2. UNIT VECTOR $\frac{\vec{a}}{|\vec{a}|}$.

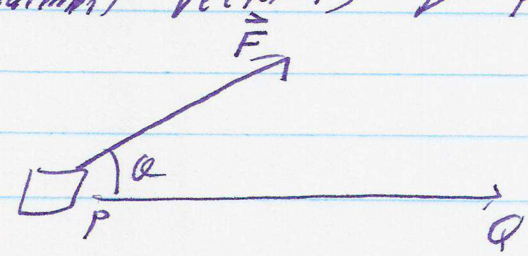
Thus $proj_a \vec{b} = comp_a \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}$ =

$$\boxed{proj_{\vec{a}} \vec{b} \quad \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \cdot \vec{a}}$$

Problems

1. Find scalar and vector projection of $\vec{b} = (1, 3, -1)$ along $\vec{a} = (2, 1, 5)$

2. Def If a force $\vec{F}$ moves an object from P to Q the displacement vector is $\vec{D} = P\vec{Q}$.



$$Work = Force \cdot Dist = |F| \cos\theta \cdot |PQ|$$

$$= F \cdot D$$

$$\boxed{W = \vec{F} \cdot \vec{D}}$$

Ex Find work done by force $F = 2\vec{i} - 6\vec{j} + 2\vec{k}$ moving object from $(1, 2, 6)$ to $(6, 12, 1)$ along st line.

distance meters
force in newtons

Joule = Newt · meter