Lecture 18 /14

So far: 1. $f: \mathbb{R} \to \mathbb{R}^n$ $f(t) = (t, \cos t, t^2+1)$ space curves

2. Real valued functions $f: \mathbb{R}^n \to \mathbb{R}$

Ex $f(x,y,z) = x^2y + yz$

Vector calculus

3 $f: \mathbb{R}^n \to \mathbb{R}^m$ vector valued functions

Ex $f(x,y) = (x+y, y^2)$

each point is assigned a vector

Ex $f: \mathbb{R}^n \to \mathbb{R}$ then $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$

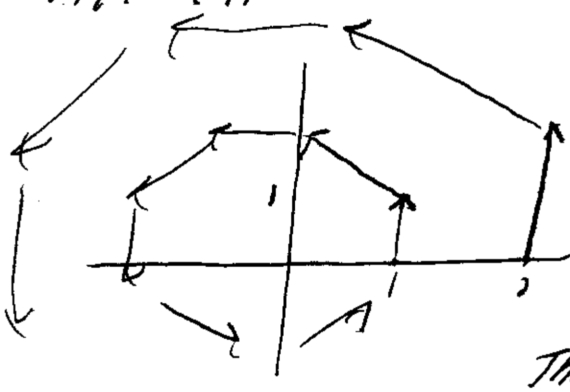
f as above $\nabla f = (2xy, x^2+z, y)$

Def When $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ we call $\vec{F}$ a vector field, each point is assigned a vector, can represent by drawing vectors

Ex $\vec{F}(x,y) = (-y, x)$

$F(1,0) = (0,1)$ $F(1,1) = (-1,1)$

$F(2,0) = (0,2)$ $F(0,1) = (-1,0)$

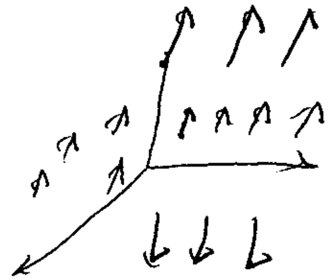


Think: rotating ccwise faster away from $(0,0)$

Rmk $(x,y) \cdot \vec{F}(x,y) = (x,y) \cdot (-y,x) = 0$

Always $\perp$

Ex. $F(x,y,z) = (0,0,z)$

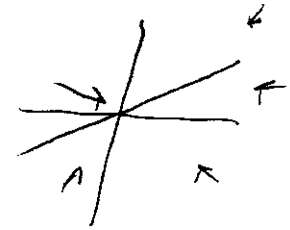


Rmk Computer very useful for sketching vector fields!

Ex. $\vec{F}(x) = \frac{-mMG}{|\vec{x}|^3}\vec{x}$ gravitational force field

$$= \left(\frac{-mMG\,x}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMG\,y}{(\;)}, -mMG\,z \right.$$

$|\vec{F}(x)| \to \infty$ as $\vec{x} \to 0$



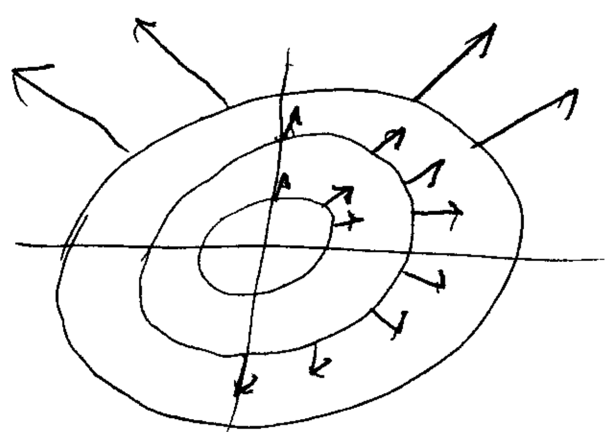
A special Case

Whenever $f: \mathbb{R}^n \to \mathbb{R}$ is a function then ∇f is a vector field

Ex $f(x,y) = x\sin(xy)$ $\quad \nabla f(x,y) = (\sin(xy) + xy\cos(xy), x^2\cos(xy))$

Recall ∇f is always $\perp$ to level curves of f.

Ex $F(x,y) = 6x^2+y^2$ $\quad \nabla F = (12x, 2y)$



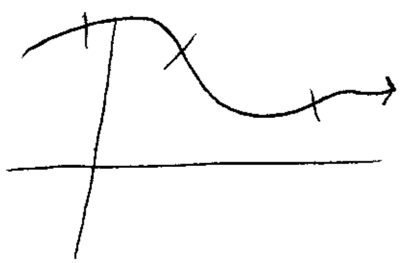
DEF A vector field $\vec{F}$ is <u>conservative</u> if $\vec{F} = \nabla f$ for some f. f is called a <u>potential function</u> for $\vec{F}$

Ex $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$ $\quad \nabla f =$ Grav Force field

Ex $\vec{F} = (e^{xy} + xye^{xy},\ x^2 e^{xy})$ is conservative with potential function $f(x,y) = xe^{xy}$.

Line Integrals

- Goal: Integrate a function "over a curve"



$C = (x(t), y(t)) \quad a \le t \le b$

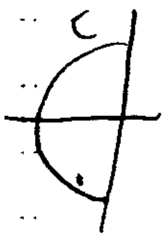
1. Divvy $[a,b]$ into pieces
2. Let $(x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))$
3. Add up

Def $\displaystyle\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,ds$

Rmk $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

line integral w.r.t. arc length

Ex Do $\int_C xy^2\,ds$ where C is left side of circle of radius 2 at center (0,0)



$$x(t) = (2\cos t, 2\sin t) \quad \pi/2 \le t \le 3\pi/2$$
$$ds = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

$$\int_{\pi/2}^{3\pi/2} 2\cos t \cdot 4\sin^2 t \cdot 2\,dt = \int_{\pi/2}^{3\pi/2} 16\sin^2 t\cos t\,dt$$
$$= 16/3 \sin^3 t \Big|_{\pi/2}^{3\pi/2}$$
$$= 16/3(-1-1) = -32/3$$

Ex $\int \sin x\,dx + \cos y\,dy$ C = top half unit circle from (1,0) to (-1,0) the line segment (-1,0) to (2,3)

Line integrals w.r.t. x or y

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Ex $\int y^2 dx + x\,dy$ 1. C = arc of parabola $x = 4 - y^2$ from (-5,-3) to (0,2).
2. C = line segment same points

Curve in 3-space $\int_C f(x,y,z)\,ds = \int f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$

General Formula

curve $\vec{r}(t)$; $a \le t \le b$

$$\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_C f(x,y,z)\,ds$$

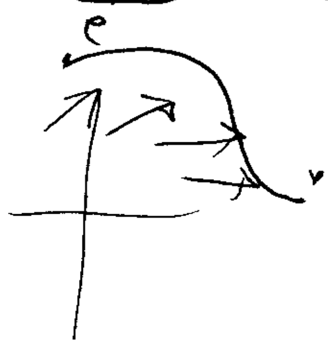
Rmk Special case $A.L. = \int_a^b |\vec{r}\,'(t)|\,dt$

Ex 1. $\int_C 2x+5z\,ds \quad C = (t, t^2, t^3) \quad 0 \le t \le 1$

2. $\int_C 2x+5z\,dz$

Recall Work $= \vec{F}\cdot\vec{D}$

Problem Calculate work along a ~~vo~~ path in a vect field $\vec{F}$



add up $\vec{F}\cdot$ unit tangent

$$W = \int_C \vec{F}\cdot\vec{T}\,ds$$

$$= \int_a^b \vec{F}(r(t))\cdot\frac{\vec{r}\,'(t)}{|r'(t)|}\cdot|r'(t)|\,dt$$

$$= \int_a^b \vec{F}(r(t))\cdot\vec{r}\,'(t)\,dt$$

DEF ~~A~~ F a continuous vector field defined on C given by $\vec{r}(t)$ $a \le t \le b$ then $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(r(t))\cdot\vec{r}\,'(t)\,dt$

Ex. Find work done by $F(x,y) = (x^2, -xy)$ moving a particle along quarter circle from $(1,0)$ to $(0,1)$

Rmk. $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$ ← opp dir.

Final Observation Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

Then $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

Problems

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