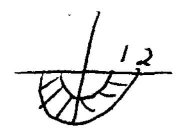
Lecture 16

Review To calculate $\iint_D f(x,y)\,dx\,dy$ using polar coord:

- put D in terms of polar
- sub in $x = r\cos\theta$ $y = r\sin\theta$
- $dx\,dy \rightarrow r\,dr\,d\theta$

Ex Calculate $\iint_D \sin(x^2+y^2)\,dx\,dy$ where $D=$ 

so $1 \le r \le 2$ $\pi \le \theta \le 2\pi$

$$\int_{\pi}^{2\pi}\int_{1}^{2} \sin(r^2)\, r\,dr\,d\theta = \int_{\pi}^{2\pi} -\tfrac{1}{2}\cos(r^2)\Big|_1^2 \, d\theta$$

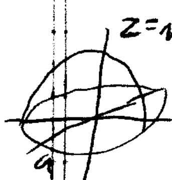
$$= \int_{\pi}^{2\pi} -\tfrac{1}{2}(\cos 4 - \cos 1)\, d\theta$$

$$= -\tfrac{1}{2}(\cos 4 - \cos 1)\,\theta \Big|_{\pi}^{2\pi}$$

$$= -\pi/2\,(\cos 4 - \cos 1)$$

Ex Find volume of sphere of radius a. $(x^2+y^2+z^2=a^2)$

$z=\sqrt{a^2-x^2-y^2}$



$$V = 2\cdot \iint_D \sqrt{a^2-x^2-y^2}\, dA = 2\cdot \int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$\vdots$$

$$= \tfrac{4}{3}\pi a^3$$

Applications of Double integrals

1. Suppose $\rho(x,y)$ = density at point (x,y) units $\frac{\text{mass}}{\text{area}}$

Then $\iint_D \rho(x,y)\,dA$ = mass of D

Similarly if $\sigma(x,y)$ = charge density then $\iint_D \sigma(x,y)\,dA$ = total charge.

2. Center of Mass

Suppose region D has density $\rho(x,y)$ (so mass $m = \iint_D \rho(x,y)\,dA$)

Let $\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \qquad \bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$

Then $(\bar{x},\bar{y})$ is the center of mass

Example

Suppose D is triangle enclosed by $x=0, y=x, 2x+y=6$, $\rho(x,y)=x^2$. Find mass & center of mass

3. Probability density function of two variables

$$f(x,y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x,y)\,dA = 1$$

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

* Understand HW problems sufficient for Exam 15.5 *

TRIPLE INTEGRALS

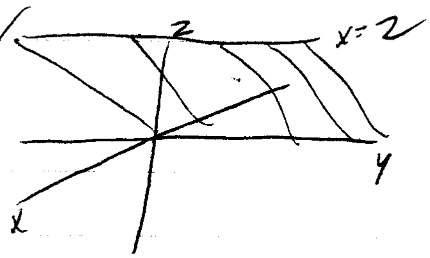
Box $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

Divide into sub boxes vol $\Delta x \Delta y \Delta z$

- $\iiint_B f(x,y,z)\, dV = \lim_{l,m,n \to \infty} \sum \sum \sum f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$
- Fubini Still Works
  - 6 ways to order

Example

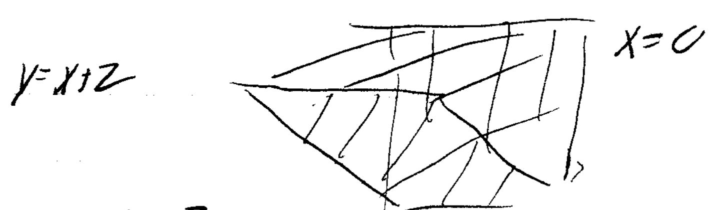
1. $\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$ What is region



$0 \le y \le x+z$

$0 \le x \le z$

$0 \le z \le 1$



$= \int_0^1 \int_0^z 6xyz \Big|_{y=0}^{x+z}\, dx\, dz$

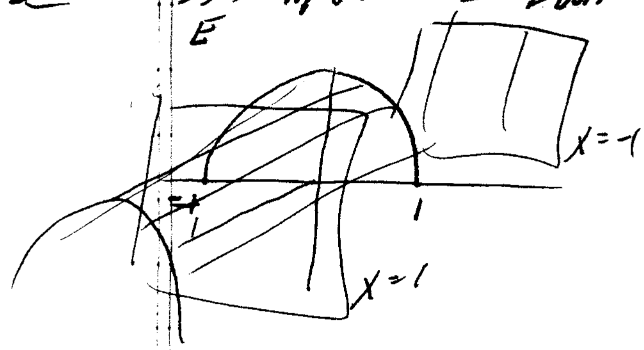
$= \int_0^1 \int_0^z 6x(x+z)z\, dx\, dz = \int_0^1 \int_0^z 6x^2z + 6xz^2\, dx\, dz$

$= \int_0^1 2x^3z + 3x^2z^2 \Big|_0^z\, dz$

$= \int_0^1 2z^4 + 3z^4\, dz = \int_0^1 5z^4\, dz = z^5 \Big|_0^1$

$= 1$

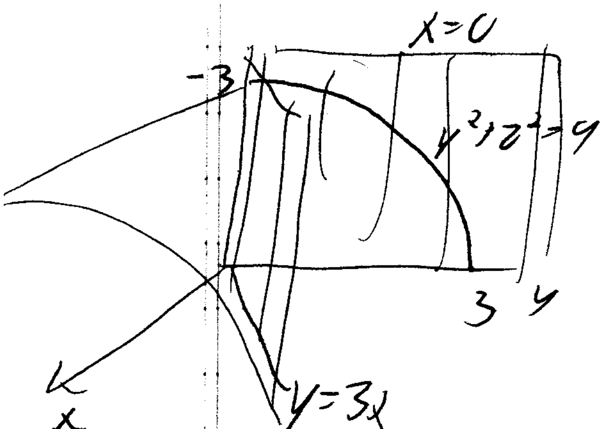
Ex $\iiint_E xy\,dV$ E bounded by $z=1-y^2$, $z=0$, $x=1$, $x=-1$

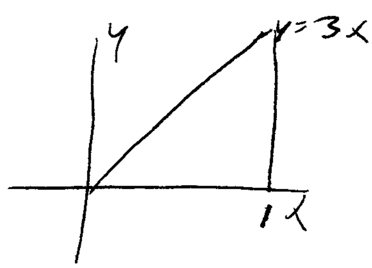


$-1 \le x \le 1$
$-1 \le y \le 1$
$0 \le z \le 1-y^2$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} xy\,dz\,dx\,dy$$

Ex $\iiint_E z\,dV$ E bounded by $y^2+z^2=9$
planes $x=0$, $y=3x$, $z=0$ in 1st octant

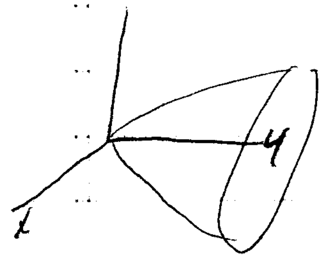


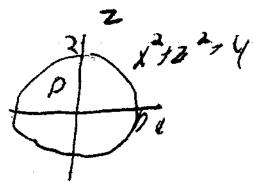


$0 \le x \le 1$
$3x \le y \le 3$
$0 \le z \le \sqrt{9-y^2}$

$$\boxed{\int_{0}^{1}\int_{3x}^{3}\int_{0}^{\sqrt{9-y^2}} z\,dz\,dy\,dx}$$

Ex. $\iiint_E \sqrt{x^2+z^2}\, dV$ E bounded by $y = x^2+z^2$, $y=4$





$x^2+z^2 \le y \le 4$

$-\sqrt{4-z^2} \le x \le \sqrt{4-z^2}$

$-2 \le z \le 2$

$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dx\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} y\sqrt{x^2+z^2}\, \Big|_{x^2+z^2}^{4}$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\, dx\, dz$$

Use polar $x = r\cos\theta$ $z = r\sin\theta$

$$= \int_0^{2\pi} \int_0^2 (4-r^2)\, r \cdot r\, dr\, d\theta \cdots = \frac{128\pi}{15}$$

Rmk Applications As before

Ex. $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$ Write in Five other ways