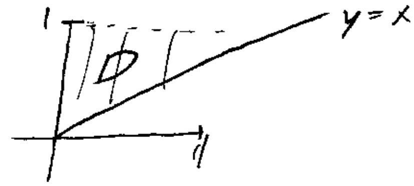
Lecture 15

Review iterated integrals.

Ex $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$



Hard integral!

Rewrite D: $0 \le y \le 1$ $0 \le x \le y$

$$= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 x\sin(y^2)\Big|_{x=0}^{x=y} dy = \int_0^1 y\sin(y^2)\,dy$$

$$= -\tfrac{1}{2}\cos(y^2)\Big|_0^1$$

$$= -\tfrac{1}{2}\cos(1) - -\tfrac{1}{2}$$

$$= \tfrac{1}{2}\left(1-\cos(1)\right)$$

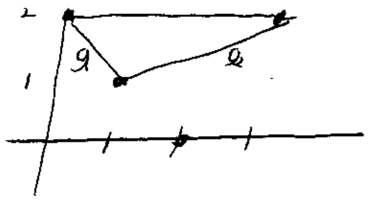
Properties of Double integrals

1. $\iint f \pm g \, dA = \iint f\,dA \pm \iint g\,dA$
2. $\iint cf\,dA = c\iint f\,dA$
3. If $D = D_1 \cup D_2$ where D_1, D_2 don't overlap except on boundary then $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

4. $\iint_D 1\,dA = \text{area}(D)$

Problems

1. $\iint y^3\,dA$ D is triangle $(0,2)$, $(1,1)$, $(3,2)$



Let $g_1(y) \le x \le g_2(y)$
$1 \le y \le 2$

$g_1:\ y = -x+2$ so $x = 2-y$

$g_2\ \ y = \frac{1}{2}x + \frac{1}{2}$ so $x = 2y-1$

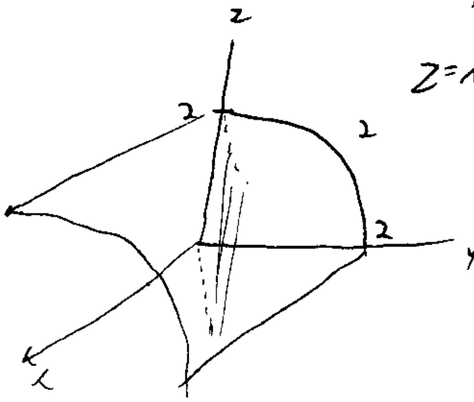
$$\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 y^3 x \Big|_{x=2-y}^{x=2y-1} dy = \int_1^2 y^3(3y-3)\,dy$$

$$= \int_1^2 3y^4 - 3y^3\,dy$$

$$= \frac{3}{5}y^5 - \frac{3}{4}y^4 \Big|_1^2$$

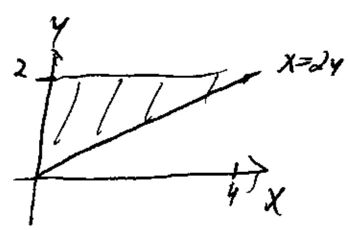
$$= \left(\frac{3}{5}\cdot 32 - 12\right) - \left(\frac{3}{5} - \frac{3}{4}\right)$$

Ex Find volume bounded by cylinder $y^2+z^2=4$ and planes $x=2y$, $x=0$, ~~y~~ $z=0$ in 1^{st} octant



$z = \sqrt{4-y^2}$

D:



$0 \le x \le 2y$
$0 \le y \le 2$

$$\int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dx\,dy$$

<u>Rmk</u> Doing it other way much harder!

$$= \int_0^2 x\sqrt{4-y^2}\Big|_{x=0}^{x=2y} dy = \int_0^2 2y\sqrt{4-y^2}\,dy = -\frac{2}{3}(4-y^2)^{3/2}\Big|_0^2$$

$$= -\frac{2}{3}(0-8) = \frac{16}{3}$$

Ex. Sketch region of integration and change order of integration

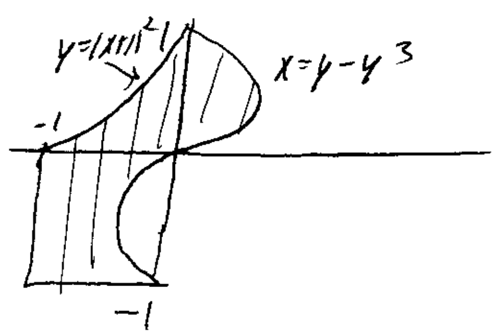
$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx \qquad \int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$$

Ex. $\iint_D y^2 e^{xy}$ $\quad D = \begin{matrix} 0 \le y \le 4 \\ 0 \le x \le y \end{matrix}$

Ex. Find volume under $z = 2x + y^2$ and above region bounded by $x = y^2$ and $x = y^3$

Ex. ~~Express~~ Evaluate $\iint_D y\,dA$

$D =$

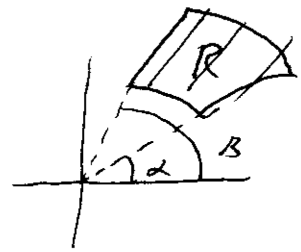


Ex. Evaluate $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\;dx\,dy$ by reversing order.

Polar coordinates

Recall $r^2 = x^2+y^2$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

Polar Rectangle $\quad a \le r \le b \quad \alpha \le \theta \le \beta$



Area R

Area $= \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{r^2\theta}{2}$

So area R $= \frac{b^2(\beta-\alpha)}{2} - \frac{a^2(\beta-\alpha)}{2} = \underbrace{\frac{1}{2}(b+a)}_{r^* \text{ center}}(b-a)\,\Delta\theta$

$= r^* \, dr\, d\theta$

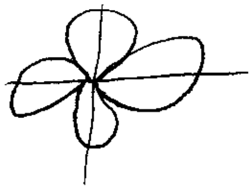
Theorem

Suppose f continuous on $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$ for $0 \le \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

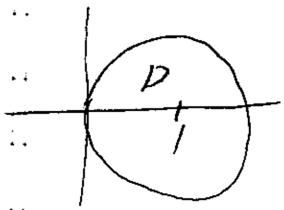
Example Find volume under $1 - x^2 - y^2 = z$ and above plane $z = 0$

Ex Find area in loop of 4-leafed rose $r = \cos 2\theta$



Ex Find volume under paraboloid $z = x^2+y^2$, above xy plane inside $x^2+y^2=2x$.

A: $(x-1)^2+y^2=1$



Want $\iint_D x^2+y^2\,dA$

$x^2+y^2=2x \to r^2 = 2r\cos\theta \to r = 2\cos\theta$

$D: \{(r,\theta) \mid -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}$

$$V = \int_{-\pi/2}^{\pi/2} \int_{\cancel{2\cos\theta}\ 0}^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$

Ex $\displaystyle\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$

Ex Volume inside $x^2+y^2=4$ and $4x^2+4y^2+z^2=64$