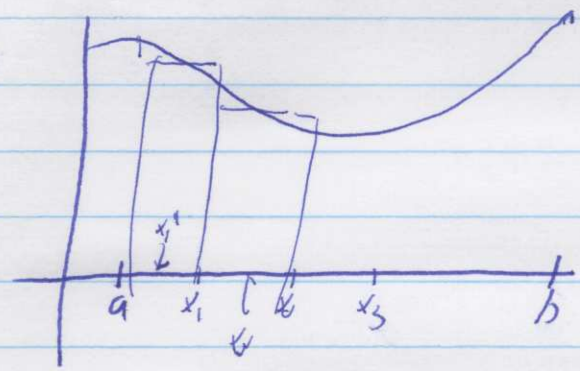
Lecture 14

Review Area under curve



$\Delta x = \frac{b-a}{n}$ n rectangles

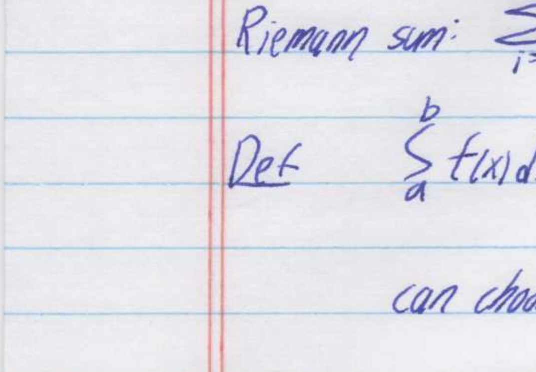
Riemann sum: $\sum_{i=1}^{n} f(x_i^*)\Delta x$

Def $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ ← measures (signed) area.

can choose x_i^* = left side, right side, midpoint, etc..

midpoint rule

Problem Find volume under a surface and above a rectangle

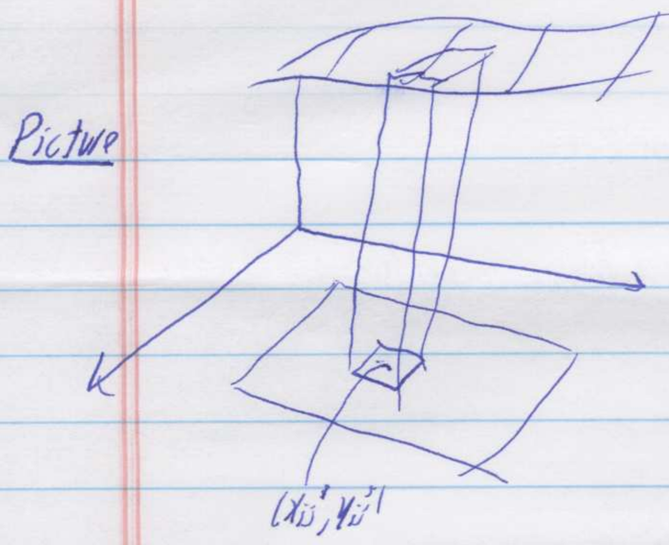


Step 1 Divide R into rectangles
[a,b] into m intervals width $\Delta x = \frac{b-a}{m}$
[c,d] into n intervals width $\Delta y = \frac{d-c}{n}$
So each has area $\Delta A = \Delta x \Delta y$

Step 2 Choose sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}

Step 3 Volume is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$

Picture



so Volume $\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$

Def The double integral of f over R is

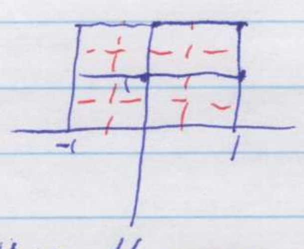
$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta X$$

if limit exists.

If so f is called integrable. As before, can choose $(x_{ij}^*, y_{ij}^*) = (x_i, y_j)$ or midpoint or . . . = midpoint rule

Fact If $f(x,y) \geq 0$ then $\iint_R f(x,y)\, dA$ measures volume above R and below surface $z = f(x,y)$

Example $f(x,y) = 20 - x^2 - y^2$ above $[-1,1] \times [0,2]$.
Estimate w/ 4 squares and upper right.



$V \approx f(0,1) \cdot 1 + f(1,1) \cdot 1 + f(0,2) \cdot 1 + f(1,2)$
$= 19 + 18 + 16 + 15 = 68$

Using 16 squares (red) $f(-\tfrac{1}{2}, \tfrac{1}{2}) \cdot \tfrac{1}{4} + f(0, \tfrac{1}{2}) \cdot \tfrac{1}{4} + \ldots$

As # rectangles $\to \infty$ then $\approx V \to$ real V.

Recall $\frac{1}{b-a}\int_a^b f(x)\,dx$ gives average value of $f(x)$ on $[a,b]$.

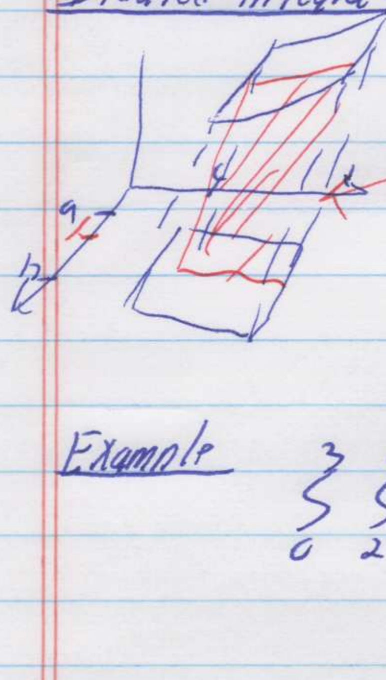
Def $f_{ave} = \frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA$ is <u>average value</u> of $f(x,y)$ over R.

Example Use midpoint rule to estimate $\iint_{[1,3]\times[0,4]} x+y^2\,dA$ using $m=n=2$.

Ex Same kind of problem with contour map or table of values.

Review In calc I we use FTOC to evaluate integrals rather than the Riemann Sum

<u>Iterated integrals</u>



$\int_c^d f(x,y)\,dy$ gives area for fixed x

Now do $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$

<u>Example</u>

$$\int_0^3 \int_2^4 xy^2\,dx\,dy = \int_0^3 \left(\frac{x^2}{2}y^2 \right)\Big|_{x=2}^{x=4} dy$$

$$= \int_0^3 8y^2 - 2y^2\,dy = 2y^3\Big|_0^3 = 54$$

$$\int_2^4 \int_0^3 xy^2\,dy\,dx = \int_2^4 \left(x\frac{y^3}{3}\right)\Big|_{y=0}^{y=3} dx = \int_2^4 9x\,dx = \frac{9}{2}x^2\Big|_2^4$$

$$= \frac{9}{2}(12) = 54$$

<u>Fubini's Thm</u> Suppose $f(x,y)$ is continuous on $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$ Then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

<u>Rmk</u> Holds for f if discontinuous on finitely many curves

<u>Example</u>

Calculate $\iint_R x\sin xy\,dA$ $\quad R=[0,1]\times[0,\pi]$

$$= \int_0^1 \int_0^\pi x\sin(xy)\,dy\,dx = \int_0^1 -\cos(xy)\Big|_{y=0}^{\pi}\,dx$$

$$= \int_0^1 -\cos(\pi x)+1\,dx$$

$$= \left[-\tfrac{1}{\pi}\sin(\pi x)+x\right]_0^1$$

$$= \boxed{1}$$

<u>Example</u> Find volume under paraboloid $z=9-x^2-y^2$ above $[0,1]\times[0,2]$

$$V = \int_0^1 \int_0^2 9-x^2-y^2\,dy\,dx = \int_0^1 \left[9y-x^2y-\tfrac{1}{3}y^3\right]_0^{y=2}\,dx$$

$$= \int_0^1 (18-2x^2-8/3)-(0)\,dx$$

$$= \left(18x-\tfrac{2}{3}x^3-\tfrac{8}{3}x\right)\Big|_0^1$$

$$= 18-\tfrac{2}{3}-\tfrac{8}{3} = \boxed{44/3}$$

Example
$$\int_0^1\int_0^3 x\sin y\,dy\,dx = \int_0^1 x \underbrace{\int_0^3 \sin y\,dy}_{\text{a number!}}\,dx$$
$$= \int_0^3 \sin y\,dy \int_0^1 x\,dx$$

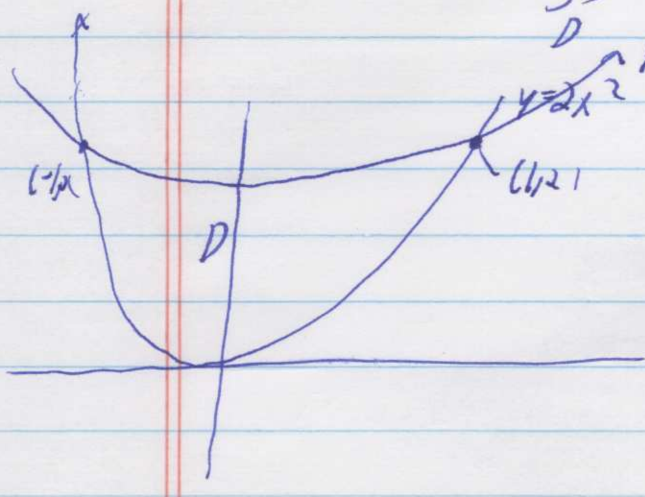
Fact If $f(x,y)=g(x)h(y)$ then $\iint_R f(x,y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$

where $R=[a,b]\times[c,d]$

Warning: Must review integration!! (u-substitution, etc...)

Problem Let D be region bounded by $y=2x^2$ and $y=1+x^2$.

Evaluate
$$\iint_D x+y^2\,dA$$



D: $-1\le x\le 1$

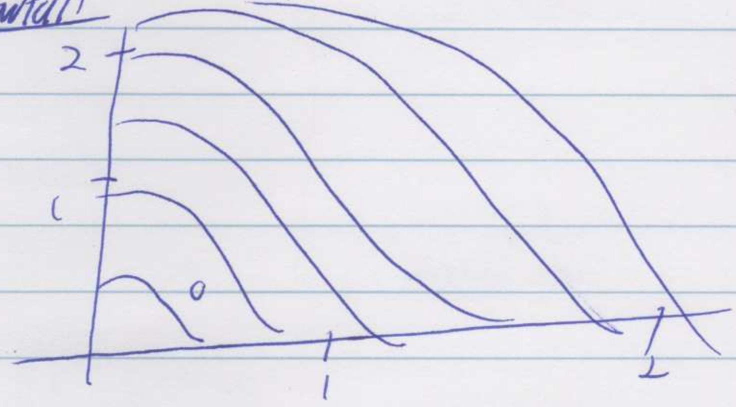
$2x^2\le y\le 1+x^2$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+y\,dy\,dx$$
$$= \int_{-1}^{1}\left[xy+\frac{y^2}{2}\right]_{2x^2}^{1+x^2}dx$$
$$= \int_{-1}^{1}\left(x(1+x^2)+\frac{(1+x^2)^2}{2}\right)-\left(x\cdot 2x^2+\frac{(2x^2)^2}{2}\right)dx$$
$$= \dots$$

D called type I region
$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$$

15.1 Problems

1. Snowfall



Estimate average snowfall

2. 20 x 30 ft swimming pool Depth

	0	5	10	15	20	25	30
0							
5							
10							
15							
20							

3. $\iint_R \sqrt{9-y^2}$ $R = [0,4] \times [0,2]$ Sketch solid

4. $\iint_R (4-2y)\,dA$ $R = [0,1] \times [0,1]$