Lecture 13  

Review Calc I max/min crit points, endpoints, B, 2nd Der Test

Given \( f(x,y) \), say \( f \) has a local maximum at \( (a,b) \) if \( f(x,y) \leq f(a,b) \) for all \( (x,y) \) near \( (a,b) \) (i.e. in little circle around \( (a,b) \)).

Then \( f(a,b) \) is local maximum value.

Similarly local minimum, global (aka absolute) max, min.

\[ \text{local max} \rightarrow \text{global max} \]

\[ \text{local min} \rightarrow \text{global min} \]

\[ \text{neither!} \]

\[ \text{Example: } f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{Theorem: If } f(x,y) \text{ has local max or local min at } (a,b) \]
\[ \text{Then } f_x(a,b) = 0 \text{ and } f_y(a,b) = 0. \text{ (necessary condition)} \]

Def: \( (a,b) \) is a critical point if \( f_x(a,b) = f_y(a,b) = 0 \), or if are DNE.

Can also write \( \nabla f(a,b) = 0 \) or \( \nabla f(a,b) \text{ DNE} \).

Ex: \( f(x,y) = 2y^2 - x^2 \). Find critical points. Find extreme value.

\[ \text{Assume } f_x = -2x \text{ and } f_y = 4y \text{ so } (0,0) \text{ only CP.} \]

Note that \( f(0,0) \geq 0 \), \( f(x,0) \leq 0 \) so not a local max or min. Called a saddle point.
\[ f_{xy} = 2y^2 - x^2 \]

(\text{crit point})

(saddle)

Second Derivative Test:
Suppose second partial derivatives of \( f \) are continuous near \((a,b)\) and \((a,b)\) is a critical point. Let

\[ D(a,b) = [f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2]. \]

a. \( D > 0 \) and \( f_{xx}(a,b) > 0 \) then local min at \((a,b)\)

b. \( D > 0 \) and \( f_{xx}(a,b) < 0 \) then local maximum

c. \( D < 0 \) then saddle point

\[ D = 0 \Rightarrow \text{no information}, \]

Remark \[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \]
by Clairaut's Theorem

Ex: Find and classify all crit points of \( f(x,y) = x^3 - 3xy^2 + y^2 \)

\[ f_x = 3x^2 - 3y^2 \quad f_y = -6xy + 2y \]

0 = 3(x-1)(x+1) \quad 0 = 2y(-3x+1)

Need \( y = \pm x \) \quad \Rightarrow y = 0 \text{ or } x = \frac{1}{3}

So \( \text{CP} \left( 0,0 \right), \left( \frac{1}{3}, -\frac{1}{3} \right), \left( \frac{1}{3}, \frac{1}{3} \right) \)
\[ f_{xx} = 6x \quad f_{yy} = -6x + 2 \quad f_{xy} = f_{yx} = -6y \]

\[ D = (6x)(6x+2) - 36y^2 \]

\[ D(0,0) = 0 \quad \text{no info} \quad \text{(can see also a saddle)} \]
\[ D(1/3, -1/3) = -4 \quad \text{and } f_{xx} > 0 \quad \text{so saddle} \]
\[ D(1/3, 1/3) = -4 \]

---

**Example:** Rect box, no lid, volume 12m$^3$ and board. Find max volume.

Given \( 2xy + 2xyz = 12 \)

\[ V = xy^2 \]

\[ V_x(y) = xy \quad \frac{12-xy}{2x+2y} \]

\[ \frac{dV}{dx} = y^2(12-2xy-x^2) \quad \frac{dV}{dy} = x^2(12-2xy-y^2) \]

Set both = 0. Clearly \( x=0 \) or \( y=0 \) are not good so

\( 12-2xy-x^2 = 0 \quad 12-2xy-y^2 = 0 \)

So \( x^2 = y^2 \) so \( x = y \) (we knew \( x + y > 0 \))

\( 12 - 2x^2 - x^2 = 0 \quad x = 2 \rightarrow y = 2 \rightarrow z = 1 \)
Theorem

Suppose \( f \) is continuous on a closed bounded set in \( \mathbb{R}^2 \).

Then:

1. \( f(x,y) \) attains an absolute max and absolute min value on \( D \).

2. The max and mins occur at critical points or on the boundary.

Example

Let \( f(x,y) = 3 + xy - x - 2y \). Let \( D \) be closed triangle w/ vertices \((0), (2,0), \text{ and } (1,3)\). Find global extrema.

Example. As above for \( f(x,y) = xy^2 \)

\[ D = \{ (x,y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3 \} \]

Example. Show \( f(x,y) = -(x^2-1)^2 - (x^2y - x - 11)^2 \) has 2 C.P. both local maxima.

Ex. \( f(x,y) = e^{x+y} \) Find local max/mins

Ex. Find points on \( y^2 = 9 + x^2 \) closest to origin