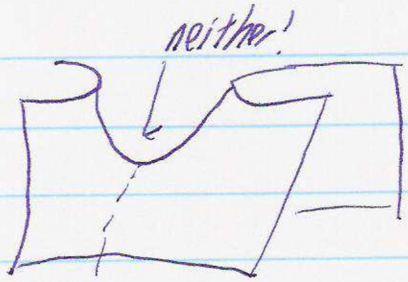
Lecture 13 • Review Calc I max/min crit points, endpoints, 2nd Der Test

Given $f(x,y)$, say f has a *local maximum* at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) "near (a,b)" (i.e. in little circle around (a,b)).
Then $f(a,b)$ is local maximum value.
Similarly local minimum, Global (aka absolute) max, min

Ex

Ex



Theorem If $f(x,y)$ has local max or local min at (a,b)
Then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. (necessary condition)

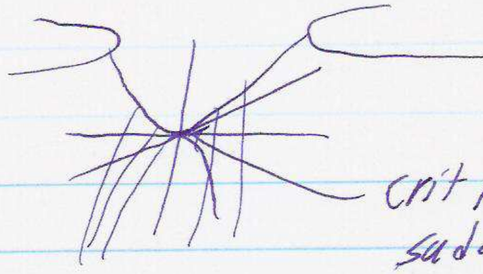
Def (a,b) is a *critical point* if $f_x(a,b) = f_y(a,b) = 0$, or if one DNE
Can also write $\nabla f(a,b) = 0$, or $\nabla f(a,b)$ DNE.

Ex $f(x,y) = 2y^2 - x^2$. Find critical points. Find extreme values

Answer $f_x = -2x$ $f_y = 4y$ so $(0,0)$ only C.P.

Note that $f(0,y) \ge 0$, $f(x,0) \le 0$ so Not a local max or min. Called a saddle point.

$f(x,y) = 2y^2 - x^2$



crit point, saddle

Second Derivative Test

Suppose second partial derivatives of f are continuous near (a,b) and (a,b) is a critical point. Let

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2.$$

a. $D>0$ and $f_{xx}(a,b)>0$ then local min at (a,b)
b. $D>0$ and $f_{xx}(a,b)<0$ then local maximum
c. $D<0$ then saddle point

$D=0 \Rightarrow$ NO INFORMATION,

Remark $\quad D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ by Clairaut Thm

Ex Find and classify all crit points of $f(x,y) = x^3 - 3xy^2 + y^2$

$f_x = 3x^2 - 3y^2 \qquad f_y = \cancel{3x} - 6xy + 2y$

$0 = 3(x-y)(x+y) \qquad 0 = 2y(-3x+1)$

Need $y = \pm x \qquad \to y=0$ or $x = 1/3$

So C.P. $(0,0)$, $(1/3,-1/3)$, $(1/3,1/3)$

$f_{xx} = 6x \quad f_{yy} = -6x + 2 \quad f_{xy} = f_{yx} = -6y$

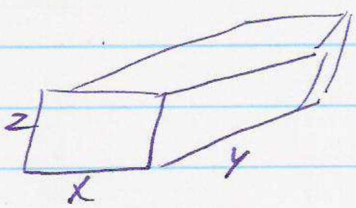
$D = 6x(-6x+2) - 36y^2$

$D(0,0) = 0$ no info (can see also a saddle)

$D(1/3, -1/3) = -4$ and ~~$f_{xx} = 2$ so~~ so saddle

$D(1/3, 1/3) = -4$ " "

Ex (p.87) Rect box, no lid, volume $12m^2$ cardboard. Find max volume.



Given $2xz + xy + 2yz = 12$

$z = \dfrac{12 - xy}{2x + 2y}$

$V = xyz$

$V(x,y) = xy \dfrac{12 - xy}{2x + 2y}$

$\dfrac{dV}{dx} = \dfrac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad \dfrac{dV}{dy} = \dfrac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$

Set both $= 0$. Clearly $x = 0$ or $y = 0$ are no good so

$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$

so $x^2 = y^2$ so $x = y$ (we know $x, y \geq 0$)

$12 - 2x^2 - x^2 = 0$ $\boxed{x = 2 \to y = 2 \to z = 1}$

Def Closed set

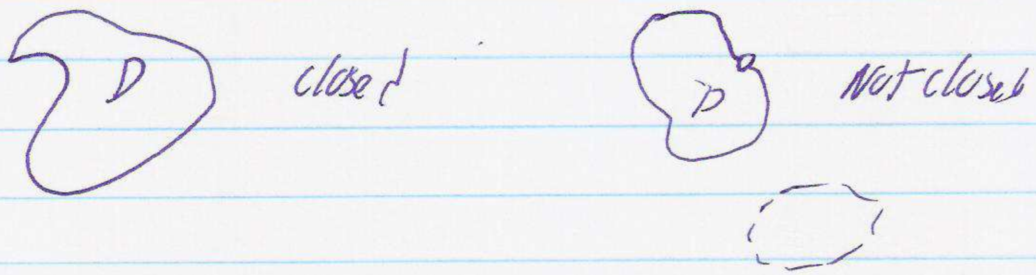
Theorem

Suppose f is continuous on a closed bounded set in $\mathbb{R}^2$.
Then:

1. $f(x,y)$ attains an absolute max and absolute min value on D.

2. The max and mins occur at critical points or on the boundary.

Ex



Example

Let $f(x,y) = 3 + xy - x - 2y$. Let D be closed triangle w/ vertices $(0,0)$, $(2,0)$ & $(0,3)$. Find global extrema.

Example. As above for $f(x,y) = xy^2$

$$D = \{ (x,y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3$$

Example Show $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$

has 2 C.P., both local maxima.

Ex $f(x,y) = e^x \cos y$. Find local max/mins

Ex Find points on $y^2 = 9 + xz$ closest to origin