Lecture 12

Review $z = f(x,y)$ then

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}, \text{ similar for } f_y.$$

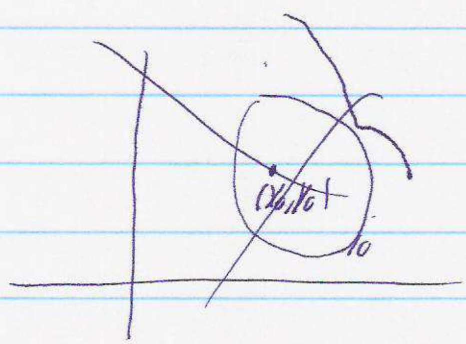
These tell rate of change in direction $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.
What about other directions?

Def Let $\vec{u} = (a,b)$ be a unit vector. The directional derivative of f at (x_0,y_0) is

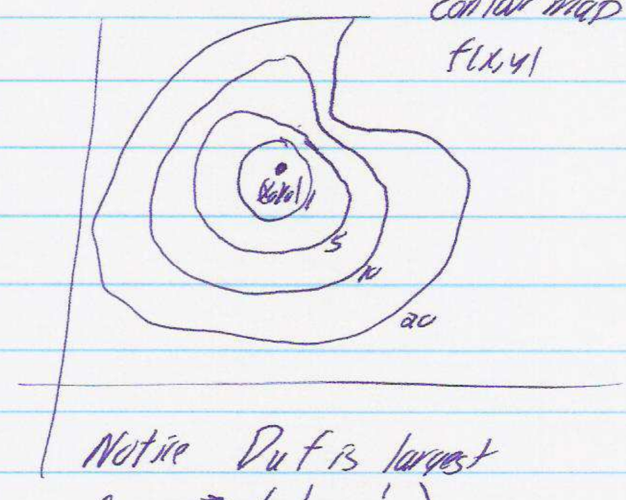
$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}.$$

Rmk $f_x = D_{\vec{i}} f, \quad f_y = D_{\vec{j}} f.$

Ex



contour map $f(x,y)$



Notice $D_u f$ is largest for $\vec{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Def Gradient $\nabla f = \langle f_x, f_y \rangle$

Thm If f is differentiable than for unit vector $\vec{u} = (a,b)$:

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

$$= \nabla f(x,y) \cdot \vec{u}$$

* To find Direct Derivative just take dot product w/ ∇f. at $\vec{u}$

Example

1. Let $f(x,y) = y + 2xy^2$. Find dir derivative at $(2,3)$ in direction of $\vec{v} = (1,2)$.

Answer: ~~f(x)~~ $\nabla f = (2y^2, 1 + 4xy)$ $\quad \vec{u} = (1/\sqrt{5}, 2/\sqrt{5})$

$\nabla f(2,3) = (18, 25)$

$D_{\vec{u}} f(2,3) = (18, 25) \cdot (1/\sqrt{5}, 2/\sqrt{5}) = \boxed{68/\sqrt{5}}$

2. Let $f(x,y,z) = e^x + ye^z + z^2$. Find Dir. Der. at $(0,0,1)$ in direction of $\vec{v} = (1,1,1)$.

A: $\nabla f = (e^x, e^z, ye^z + 2z)$ $\quad \vec{u} = 1/\sqrt{3}\,(1,1,1)$

$\nabla f(0,0,1) = (1, e, 2)$

$\nabla f(0,0,1) \cdot \vec{u} = \boxed{1/\sqrt{3}\,(3 + e)}$

Problem Which direction does $f(x,y)$ have maximum or min rate of change?

Theorem Suppose f is diff'ble. The maximum value of $D_{\vec{u}} f(\vec{x})$ occurs when $\vec{u}$ points in same direction as ∇f and this maximum value is $|\nabla f(\vec{x})|$.

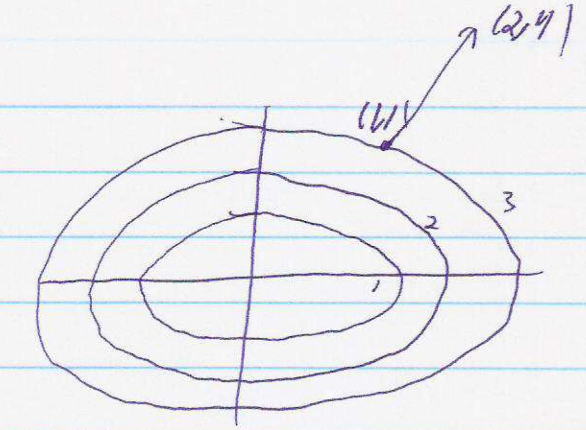
Proof $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f|\,|\vec{u}| \cos\theta = |\nabla f| \cos\theta$

is a max when $\theta = 0$

min when $\theta = \pi$.

Example

$f(x,y) = x^2 + 2y^2$



$\nabla f = (2x, 4y)$

$\nabla f(1,1) = (2,4)$

∇f points in direction of fastest increase,

$-\nabla f$ points in direction of fastest decrease,

Problem

$f(x,y) = \sin(xy) + x$ Find max rate of change of f at point $(1,0)$ and direction in which it occurs

A: $\cancel{\nabla f = (y\cos(xy),\ x\cos(xy)) \quad \nabla f(1,0) = (0,1)}$

$$\nabla f(x,y) = (1 + y\cos(xy),\ x\cos(xy)) \qquad \nabla f(1,0) = (1,1)$$

$$\boxed{\begin{array}{l}\text{Direction: } (1/\sqrt{2},\ 1/\sqrt{2}) \\ \text{max rate} = |\nabla f(0,1)| = \sqrt{2}\end{array}}$$

An important property of the gradient vector

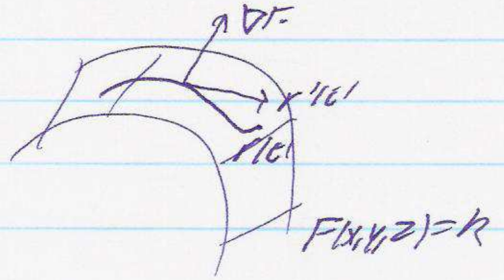
Suppose Level surface $F(x,y,z)=k$, and let (x_0,y_0,z_0) be on surface.
Suppose $\vec{r}(t)=(x(t),y(t),z(t))$ is a curve on the surface S.

$$F(x(t),y(t),z(t))=k. \quad \text{Apply } \frac{\partial}{\partial t}\text{, use Chain rule.}$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial t}=0$$

$$\nabla F \cdot \vec{r}\,'(t)=0$$

so ∇F is $\perp$ to all curves in Level surface



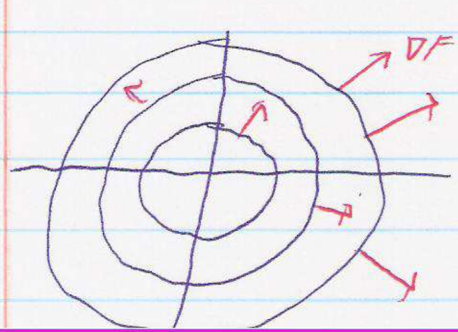
. We have prove

<u>Theorem</u> The gradient $\nabla F(x_0,y_0,z_0)$ is always perpendicular to the level surface $F(x,y,z)=F(x_0,y_0,z_0)$.

<u>COR</u> Tangent plane to level surface $F(x,y,z)=k$ at (x_0,y_0,z_0) is

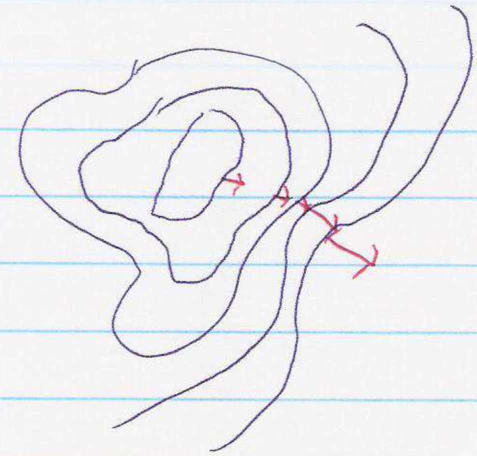
$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

<u>Example</u> $F(x,y)=(x^2+y^2)$ $\quad \nabla F=(2x,2y)$



At each pt ∇F is $\perp$ to level curves

Ex



∇f will trace out curve of steepest ascent.

computer optimization.

Problems

1. Let $f(x,y) = e^{xy}$ at $(2,0)$.
   a. Find ∇f b. Find eq of tangent plane to graph at $(2,0)$

2. $f(x,y,z) = x^2y + y^2z + z^2x$ at $(1,-1,1)$. Find tangent plane to level surface passing through this point. Find parametric equation for normal line.

3. In what directions at the point $(2,0)$ does $f(x,y) = xy$ have rate of change -1? How about -3?

4. In what direction at point (a,b,c) does $f(x,y,z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point.

5. Find rate of change of $f(x,y) = \frac{x}{1+y}$ at $(0,0)$ in the direction of $\vec{i} - \vec{j}$.

6. $z + 1 = xe^y \cos z$ at $(1,0,0)$. Find eqs of tangent plane and normal line at this point

SUMMARY

1. ∇f points in direction of maximal increase of f and is perpendicular to level curves (surfaces) $f = K$.
2. $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ for $\vec{u}$ a unit vector.