Lecture 11

Review Suppose $y = f(x)$ and $x = g(t)$ so $y = f(g(t))$

Then

$$\frac{dy}{dt} = f'(g(t))\,g'(t) \quad \underline{\text{chain rule}}$$

Alternate notation: $f' = \frac{dy}{dx}$ so $\quad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

General Chain Rule $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^k$ so $G \circ F : \mathbb{R}^n \to \mathbb{R}^k$

has n variables and k coordinate functions [calc 1: $n = m = k = 1$]

Ex $\quad F(x,y,z) = (x^2 y, x+z) \quad \mathbb{R}^3 \to \mathbb{R}^2$

$\quad G(u,v) = (u^2, u+v, v^3, uv) \quad \mathbb{R}^2 \to \mathbb{R}^4$

$\quad G(F(x,y,z)) = ((x^2y)^2, x^2y + x + z, (x+z)^3, x^2y(x+z))$

$\quad G \circ F : \mathbb{R}^3 \to \mathbb{R}^4$

Goal Formula for partial derivatives of each coordinate function with respect to each variable

Special Case 1 $z = f(x,y)$ differentiable where $x = g(t)$, $y = h(t)$ both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$t \longrightarrow (x,y) \xrightarrow{f} z$$

i.e. $\quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial z}$

Example

$$Z = \sin(xy) + x^2$$
$$x = t^3 \quad y = t^2 + t$$

Chain Rule
$$\boxed{\frac{dz}{dt} = (y\cos(xy) + 2x)(3t^2) + x\cos(xy)(2t+1)}$$

Alternately
$$Z = \sin(t^3(t^2+t)) + (t^3)^2$$
$$\boxed{\frac{dz}{dt} = \cos(t^3(t^2+t)) \cdot [3t^2(t^2+t) + t^3(2t+1)] + 2(t^3) \cdot 3t^2}$$

equal if sub in $x = t^3 \quad y = t^2 + t$

Example

$Z = x^2 + xy \qquad x = \cos t \quad y = \sin t \qquad$ Find $\frac{dz}{dt}$ at $t = \pi/2$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x+y)(-\sin t) + x\cos t$$

At $t = \pi/2 \quad x = 0 \quad y = 1$

$$\frac{dz}{dt} = 1(-1) + 0 = -1$$

$$\boxed{\left.\frac{dz}{dt}\right|_{t=0} = -1}$$

This is the rate of change of function $f(x,y) = x^2 + xy$ as a particle moves on circle $(\cos t, \sin t)$ at point $(0,1)$

Special Case 2

Suppose $z = f(x,y)$ where $x = g(s,t)$ $y = h(s,t)$.
all differentiable. The

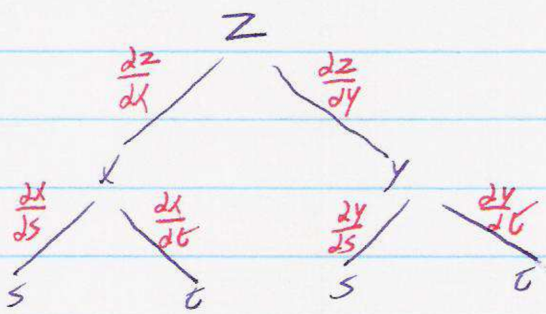
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$s,t \longrightarrow x,y \longrightarrow z$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$ so z is a function of s & t.

Tree diagram



General Chain Rule

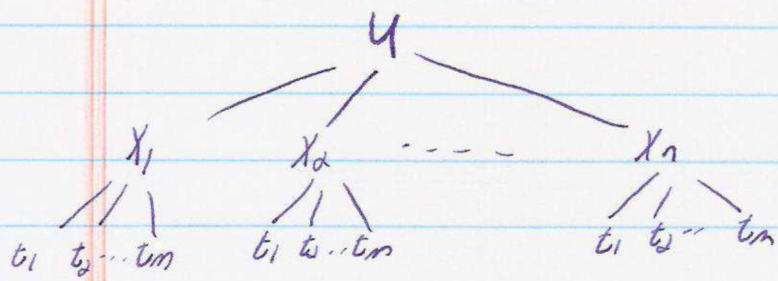
Suppose $u = f(x_1, x_2, \ldots, x_n)$ is differentiable and each x_j is a function of $t_1, t_2, \ldots, t_m$. Then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$t_1, t_2, \ldots, t_m \longrightarrow x_1, x_2, \ldots, x_n \xrightarrow{f} u$$

$$\mathbb{R}^m \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}$$

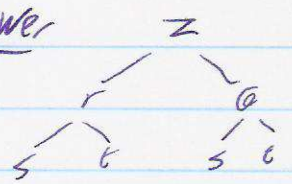
Tree diagram for general chain rule:



Ex $z = e^r \cos\theta \quad r = st, \quad \theta = \sqrt{s^2+t^2}$

Find $\frac{\partial z}{\partial t}, \quad \frac{\partial z}{\partial s}$

Answer



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$= (e^r\cos\theta)(s) + (-e^r\sin\theta)\cdot\frac{2t}{\sqrt{s^2+t^2}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= (e^r\cos\theta)t + (-e^r\sin\theta)\frac{2s}{\sqrt{s^2+t^2}}$$

Rmk To find $\left.\frac{\partial z}{\partial t}\right|_{s=1,\, t=3}$ we would plug in to get corresponding values of r, θ.

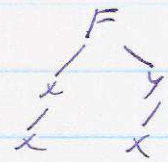
Problem $w = \ln(x^2+y^2+z^2) \quad x = t \quad y = \cos t \quad z = t^2$

Find $\frac{dw}{dt}$

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{\partial x}{\partial t} + \frac{dw}{dy}\frac{dy}{dt} + \frac{dw}{dz}\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}\,1 + \frac{2y}{x^2+y^2+z^2}(-\sin t) + \frac{2z}{x^2+y^2+z^2}(2t)$$

More implicit differentiation



Given $F(x,y)=0$. Find $\frac{\partial y}{\partial x}$. We assume $y=y(x)$.

So $$0=\frac{\partial F}{\partial x}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}$$

$$0=F_x+F_y\frac{\partial y}{\partial x}$$

$$\boxed{\frac{\partial y}{\partial x}=\frac{-F_x}{F_y}}$$

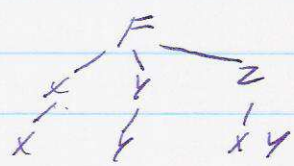
Example $y^2+xy+6x^3=5$. Find $\frac{dy}{dx}$.

$$F(x,y)=y^2+xy+6x^3-5$$

$$F_x=y+18x^2 \qquad F_y=2y+x$$

$$\boxed{\frac{dy}{dx}=\frac{-(y+18x^2)}{2y+x}}$$

Higher Dimensions Given $F(x,y,z)=0$. Assume $z=z(x,y)$



$$\boxed{\frac{\partial z}{\partial x}=\frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y}=\frac{-F_y}{F_z}}$$

Problems

1. Given $x^2+y^2+z^2=3xyz$. Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

2. $u=x^2+yz$ $\quad x=pr\cos\theta \quad y=pr\sin\theta \quad z=p+r$

   Find $\frac{\partial u}{\partial p}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ when $p=2, r=3, \theta=0$

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4. Let $z=f(x+at)+g(x-at)$ Show

$$\underbrace{\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}}_{\text{wave equation}}$$

   Hint Let $u=x+at \quad v=x-at$

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