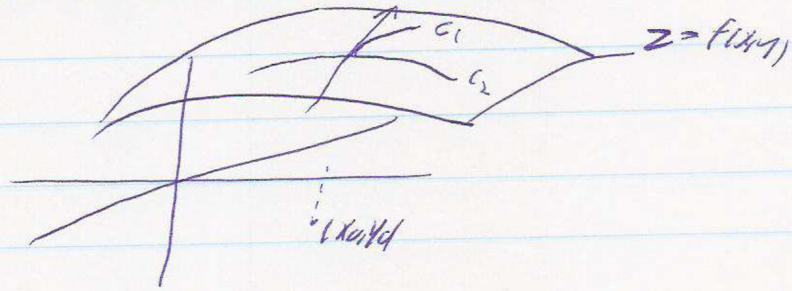
Lecture 10

Recall Partial derivatives: hold all but one variable constant.

Example $f(x,y)$



C_1 is curve $z = f(x, y_0)$ $\quad$ C_2 curve $f(x_0, y_0)$

Thus: $f_x(x_0,y_0)$ is slope of tangent line to C_1 in plane $y=y_0$
$f_y(x_0,y_0)$ is slope " " " " " " " $x=y_0$

Tangent Plane

Vectors $(0, 1, f_y(x_0,y_0)) = \vec{T}_1$
$(1, 0, f_x(x_0,y_0)) - \vec{T}_2$

$$\vec{n} = \vec{T}_1 \times \vec{T}_2 = (f_x(x_0,y_0), f_y(x_0,y_0), -1)$$

Theorem Suppose $f(x,y)$ has continuous partial derivatives.
The tangent plane to surface $z = f(x,y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Proof $\vec{n} = (f_x(x_0,y_0), f_y(x_0,y_0), -1)$

RmK $z_0 = f(x_0,y_0)$

Ex

Find eq of tangent plane to $z = 3x^2 - y^2$ at $(2, 1, 10)$

A $f_x = 6x$ $\quad f_x(2,1) = 12$

$f_y = -2y$ $\quad f_y(2,-1) = -2$

$$\boxed{z - 10 = 12(x-2) - 2(y-1)}$$

Higher Order Derivatives

Ex $(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{11}$ ← Many notations

$f_{xy} = (f_x)_y = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{12}$

Example

$f(x,y) = x^2 + xy^2 + y$

$f_x = 2x + y^2$ $\qquad f_y = 2xy + 1$

$f_{xx} = 2$ $\qquad f_{xy} = 2y$

$f_{yx} = 2y$ $\qquad f_{yy} = 2x$

Clairaut's Thm - "Equality of mixed partial derivatives"

Suppose $f(x,y)$ is defined on a disc containing (a,b) and f_{xy}, f_{yx} both continuous on D Then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

p. 891 #95 $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{else} \end{cases}$

Then

$f_{xy}(0,0) = -1$ $f_{yx}(0,0) = 1$ but f_{xy}, f_{yx} not continuous at $(0,0)$

Problem

$W = \dfrac{e^v}{v^2+u^2}$ Find partial derivatives

A: $\dfrac{\partial w}{\partial v} = \dfrac{(v^2+u^2)e^v - e^v(2v)}{(v^2+u^2)^2}$ $\dfrac{\partial w}{\partial u} = \dfrac{0 - 2ue^v}{(v^2+u^2)^2}$

Problem

$x^2+y^2+z^2 = 3xyz$ Use implicit diff to find $\frac{\partial z}{\partial x}$

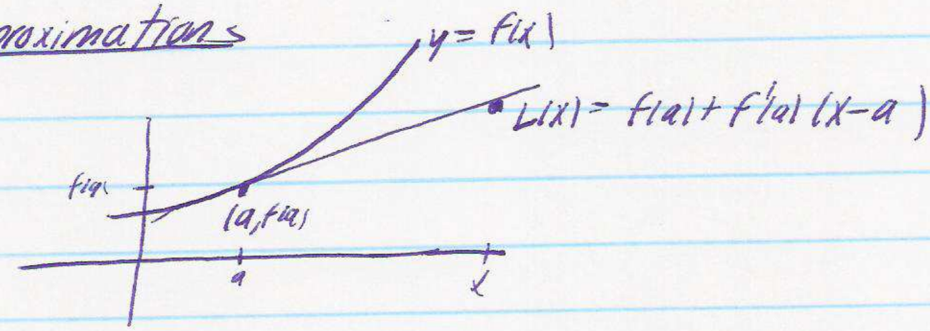
A Assume $z = f(x,y)$ w/out solving for z

$2x + 0 + 2z\frac{\partial z}{\partial x} = 3\cancel{x}yz + 3xy\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z + 3xy}$$

Linear Approximations

Review:



Equation of tangent line is $y = f'(a)(x-a) + f(a)$

$L(x) = f'(a)(x-a) + f(a)$ is a <u>linear approximation</u> to $f(x)$ for x near a.

<u>Goal</u> Tangent plane is close to graph $f(x,y)$ also.

<u>Def</u> $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

is the <u>linear approximation</u> of $f(x,y)$ at (a,b).

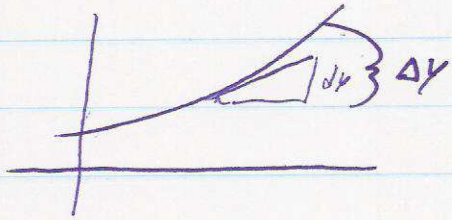
<u>Problem</u> Find linearization of $f(x,y) = \sqrt{x + e^{4y}}$ at $(3,0)$

$f_x = \frac{1}{2\sqrt{x+e^{4y}}} \cdot 1 \qquad f_y = \frac{1}{\sqrt{x+e^{4y}}} \cdot 4e^{4y}$

$f_x(3,0) = \frac{1}{4} \qquad f_y(3,0) = 1 \qquad f(3,0) = 2$

$$\boxed{L(x,y) = 2 + \tfrac{1}{4}(x-3) + y}$$

Recall $y = f(x)$ differential $dy = f'(x)\,dx$



Def $z = f(x,y)$ define

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$
$$= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Linear Approximation $L(x,y) = f(a,b) + dz$

Problems

1. Find linearization $L(x,y)$ for $f(x,y) = \sin(2x + 3y)$ at $(-3, 2)$.

   use it to approximate $f(-3.1, 2.1)$.

2. $z = x^3 \cos(y^2)$ Find dz

   $T = \dfrac{u+v}{1-uv}$ Find dT

3. A closed cylindrical can is 10cm high, 4cm diam, metal in top, bottom is 0.1cm thick, in side is 0.05cm thick. Use differentials to estimate amount of metal

4. Find tangent plane to $z = 4x^2 + 4xy$ at $(1, 3)$

5. Use contour plot to estimate partial derivatives