Lecture 8 - Partial Fractions

Def A rational function is a $\frac{\text{polynomial}}{\text{polynomial}}$, ex: $\frac{3x^2+x-1}{x^5+2x^2+x} = \frac{p(x)}{q(x)}$

Goal Integrate any rational function.

Remark 1 We only need to handle case where $\deg p(x) < \deg q(x)$, by using long division.

Ex $\frac{x^3+x^2-2x+1}{x^2-1}$

$$\begin{array}{r} x+1 \\ x^2-1 \overline{\smash{)}\, x^3+x^2-2x+1} \\ \underline{x^3 \quad\;\; -x} \\ x^2-x+1 \\ \underline{x^2 \quad\;\; -1} \\ -x+2 \end{array}$$

so $x^3+x^2-2x+1 = (x^2-1)(x+1) + -x+2$

$= x+1 + \frac{-x+2}{x^2-1}$

Key examples that we can handle already:

1. $\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b|$ **Ex** $\int \frac{3}{2x+5}\,dx = \frac{3}{2}\int \frac{2}{2x+5}\,dx$
$= \frac{3}{2}\ln|2x+5| + C$

2. $\int \frac{A}{(ax+b)^k}\,dx \quad k>1$
$= \frac{A}{a}\cdot\frac{1}{-k+1}(ax+b)^{-k+1}$

Ex $\int \frac{7}{(3x+2)^5}\,dx \quad u=(3x+2), \; du = 3dx$

$= \int \frac{7}{3}\cdot u^{-5}\,du = \frac{-7}{12}u^{-4}$

$= \frac{-7}{12(3x+2)^4} + C$

Both easily handled by u-substitution!

3. $\int \frac{Ax+B}{ax^2+bx+c}$ with ax^2+bx+c irreducible.

We can always do these by completing the square and using

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \quad \text{plus } \int \frac{1}{u}\,du$$

EX $\int \frac{x+1}{x^2-4x+8}\,dx$ $\qquad x^2-4x+8=(x-2)^2+4$

$u=x-2 \quad du=dx$

$$= \int \frac{u+3}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du + \int \frac{3}{u^2+4}\,du$$

$$= \frac{1}{2}\ln|u^2+4| + \frac{3}{2}\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= \frac{1}{2}\ln|x^2-4x+8| + \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right)+C$$

Fact: (Almost) Every $\frac{p(x)}{q(x)}$ is a sum of terms 1→3

Procedure for integrating $p(x)/q(x)$.

1. If $\deg p(x) \geq \deg q(x)$, do long division to get $\text{polyn} + \frac{\tilde{p}}{q}$.

2. Factor $q(x)$ into irreducibles - Fact: all quadratics or linear terms

3. Do partial fraction procedure.

CASE 1/2

$q(x)$ all linear terms

Ex $$\frac{2x}{(x-1)(x+2)^3(2x+3)(x+5)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} + \frac{E}{2x+3} + \frac{F}{x+5} + \frac{G}{(x+5)^2}$$

solve for A, B, C, D, E, F, G

Ex $$\frac{x^2+1}{(2x-1)^4} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(2x-1)^3} + \frac{D}{(2x-1)^4}$$

How to solve?

- clear denominators
  - plug in "strategic" values of x
  - or multiply out, equate coefs

Ex $$\int \frac{x^2+x-1}{x(2x-1)(x+2)}\,dx$$

$$\frac{x^2+x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$

Method 1

$x=0 \rightarrow -1 = 2A \qquad A = -1/2$

$x = 1/2 \rightarrow -1/4 = 5/4\,B \qquad B = -1/5$

$x=-2 \rightarrow 1 = 10C \qquad C = 1/10$

$$\boxed{\text{Answer} = -\tfrac{1}{2}\ln|x| - \tfrac{1}{10}\ln|2x-1| + \tfrac{1}{10}\ln|x+2| + C}$$

Alternately:

$$x^2+x-1 = A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$$

x^2 coef: $1 = 2A+B+2C$

x coef $1 = 3A+2B-C$

const: $-1 = -2A \cancel{-0}$ solve

Ex. $\int \frac{x}{(x-1)^2(x+1)}\,dx$

$$\frac{x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$x=1 \to 1 = 2B \to B = 1/2$

$x=-1 \to -1 = 4C \quad C = -1/4$

$x=0 \to 0 = -A+B+C \quad A = 1/4$

$$= \int \frac{1/4}{x-1} + \frac{1/2}{(x-1)^2} + \frac{-1/4}{x+1} = 1/4 \ln|x-1| + 1/2\left(\frac{-1}{x-1}\right) - \frac{1}{4}\ln|x+1| + C$$

CASE 3 If ax^2+bx+C is irreducible quadratic in denom, then add an

$$\frac{Ax+B}{ax^2+bx+C}$$ to your list.

CASE 4 (Skip) Higher powers of quadratics, e.g.

$$\int \frac{x+3}{(x^2+1)^3(x-1)^2}\,dx$$ Just as above.

Ex. $\int \frac{x+4}{(x^2+2x+5)(x+1)^2}\,dx$ $\quad \frac{x+4}{(x^2+2x+5)(x+1)^2} = \frac{Ax+B}{x^2+2x+5} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$

$x+4 = (x+1)^2(Ax+B) + C(x+1)(x^2+2x+5) + D(x^2+2x+5)$

x^3 coef: $0 = A + C$ $\quad$ so $A = -C$

x^2 coef $0 = 2A + B + 3C + D$ $\quad 0 = B + C + D$

x coef $1 = A + 2B + 2C + 5D$ $\quad 1 = 2B + C + 5D$

cons $4 = B + 5C + 5D$ $\quad 4 = B + 5C + 5D$

etc.. $B = -C - D$

$1 = -C + 3D$

$4 = 4C + 4D$

$8 = 16D$ $\quad D = 1/2$

$C = 1/2$

$B = -1$

$A = -1/2$

$\int \frac{-\frac{1}{2}x - 1}{x^2+2x+5}\,dx + \frac{1}{2}\ln|x+1| - \frac{1}{2}\cdot\frac{1}{x+1}$

$(x+1)^2 + 4$

$u = x+1 \quad x = u-1$

$\int \frac{-\frac{1}{2}u + 1/2 - 1}{u^2+1} = \int \frac{-1/2\, u}{u^2+1} - \frac{1/2}{u^2+1}\,du$

$= -\frac{1}{4}\ln|u^2+1| - 1/2\tan^{-1}(u)$

More examples

1. $$\int \frac{x^2-x+6}{x^3+3x}$$

2. $$\int \frac{x^3+4}{x^2+4}\,dx$$

3. $$\int \frac{dx}{2\sqrt{x+3}+x} \qquad u=x+3$$

4. $$\int \frac{\cos x}{\sin^2 x+\sin x}$$