Lecture 7 Inverse & Trig Substitutions

Review 1. $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$

Inverse subs

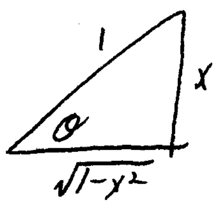
Given $\int f(x)\,dx$ substitute $x = g(t)$ $dx = g'(t)\,dt$
to get "harder" integral.

Example

$\int \sqrt{1-x^2}\,dx$ Let $x = \sin\theta$ $dx = \cos\theta\,d\theta$

$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1}{2} + \frac{\cos(2\theta)}{2}\,d\theta$

$= \theta/2 + 1/4 \sin(2\theta))$ Now $\theta = \sin^{-1} x$ $\sin 2\theta = 2\sin\theta\cos\theta$



$\cos\theta = \sqrt{1-x^2}$

Answer: $\frac{1}{2}\sin^{-1} x + \frac{1}{2} x\sqrt{1-x^2} + C$

Check $\frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2} x \cdot \frac{-2x}{2\sqrt{1-x^2}} + \frac{1}{2}\sqrt{1-x^2}$

$= \frac{1}{2\sqrt{1-x^2}} \quad \frac{-\cancel{2}x^2}{2\sqrt{1-x^2}} + \frac{\cancel{1}-x^2}{2\sqrt{1-x^2}} = \sqrt{1-x^2}$

Remark $\theta = \sin^{-1} x$ so $-\pi/2 \le \theta \le \pi/2$ so $\cos\theta \ge 0$
which justifies

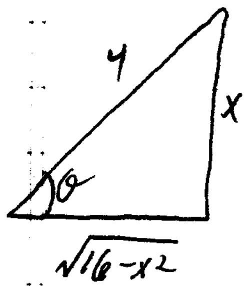
$$\sqrt{\cos^2\theta} = \cos\theta$$

Ex $\displaystyle\int \frac{\sqrt{16-x^2}}{x^2}\,dx$ Try $x = 4\sin\theta$ $dx = 4\cos\theta\,d\theta$

$$= \int \frac{4\cos\theta}{16\sin^2\theta}\cdot 4\cos\theta\,d\theta = \int \cot^2\theta\,d\theta = \int \csc^2\theta - 1\,d\theta$$

$$= -\cot\theta - \theta + C$$

Now $\theta = \sin^{-1}(x/4)$



$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Final answer: $\boxed{-\frac{\sqrt{16-x^2}}{x} - \sin^{-1}(x/4) + C}$

<u>In General</u>

- $\sqrt{a^2-x^2}$ <u>suggests</u> $x = a\sin\theta$ substitution
- $\sqrt{a^2+x^2}$ suggests $x = a\tan\theta$
- $\sqrt{x^2-a^2}$ suggests $x = a\sec\theta$

Ex $\displaystyle\int x\sqrt{4+x^2}\,dx$ $u = 4+x^2$ $du = 2x\,dx$

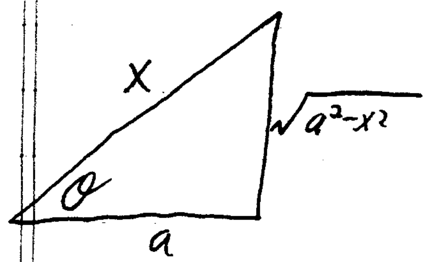
$$= \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{3}u^{3/2} = \tfrac{1}{3}(4+x^2)^{3/2} + C$$

<u>Moral</u> Don't Rush into trig substitution!

Ex $\int \frac{dx}{\sqrt{x^2-a^2}}$ $\quad x = a\sec\theta \quad dx = a\sec\theta\tan\theta\, d\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Now $\sec\theta = x/a$ $\quad$ so $\tan\theta = \frac{\sqrt{a^2-x^2}}{a}$



Answer: $\ln\left|\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{a}\right| + C$

$$= \ln|x+\sqrt{a^2-x^2}| - \ln a + C$$

$$= \ln|x+\sqrt{a^2-x^2}| + \tilde{C}$$

Completing the square

$$x^2+bx+c = (x+b/2)^2 + c - b^2/4$$

Ex $\quad x^2+x+1 = (x+1/2)^2 + 3/4$

Sometimes complete the square before doing trig sub

Ex.
$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{x}{\sqrt{(x+\frac{1}{2})^2+3/4}}\,dx \qquad u = x+1/2 \quad du = dx$$

$$= \int \frac{u-1/2}{\sqrt{u^2+3/4}}\,du = \int \frac{u}{\sqrt{u^2+3/4}}\,du - \frac{1}{2}\int \frac{1}{\sqrt{u^2+3/4}}\,du$$

$w = u^2+3/4$
$dw = 2u\,du$

$u = \frac{\sqrt{3}}{2}\tan\theta$

$= \text{etc...}$

Ex.
$$\int \frac{dx}{(x^2+2x+2)^2} \qquad x^2+2x+2 = (x+1)^2+1$$

$$= \int \frac{1}{(x+1)^2+1}\,dx \qquad u = x+1 \quad du = dx$$

$$= \int \frac{1}{u^2+1}\,du = \tan^{-1}(u)$$

$$= \tan^{-1}(x+1) + C$$

Rmk
$$\int \frac{1}{1+u^2}\,du \qquad u = \tan\theta \quad du = \sec^2\theta\,d\theta$$

$$\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int 1\,d\theta = \theta = \tan^{-1}u$$

so ok even if you forget!

Other inverse subs $\sqrt{ax+b}$ sometimes $u^2 = ax+b$
$2u\,du = a\,dx$ works

Ex $\int \frac{1}{1+\sqrt{2x}}\,dx$

$2x = u^2$
$2\,dx = 2u\,du \qquad dx = u\,du$

$$= \int \frac{1}{1+u}\,u\,du = \int \frac{u}{1+u}\,du = \int \frac{1+u-1}{1+u} = u - \ln|u+1|$$

$$= \sqrt{2x} - \ln|1+\sqrt{2x}| + C.$$

Problems

1. $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\,dt$

2. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

3. $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}$

4. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$