Lecture 6 Trigonometric Integrals

Review

1. $\tan x = \frac{\sin x}{\cos x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \csc x = \frac{1}{\sin x}$

2. $\sin^2 x + \cos^2 x = 1 \quad \xrightarrow{\frac{1}{\sin^2}} \quad 1 + \cot^2 x = \csc^2 x$

   $ \quad \xrightarrow{\frac{1}{\cos^2}} \quad \tan^2 x + 1 = \sec^2 x$

3. $\sin^2 x = \frac{1-\cos 2x}{2}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

4. $\sin(2x) = 2 \sin x \cos x$

Ex $\int \sin^{20} x \cos x \, dx$ this is easy!

$u = \sin x \quad du = \cos x \, dx \qquad \int u^{20} du = \frac{u^{21}}{21} = \frac{\sin^{21} x}{21}$

Ex $\int \sin^5 x \, \cos^3 x \, dx$

Goal! Leave one cosine for our du!

$= \int \sin^5 x (1 - \sin^2 x) \cos x \, dx$

$= \int \sin^5 x \cos x - \sin^7 x \cos x \, dx$

$= \frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + C$

Ex $\int \sin^5 x \cos^2 x \, dx$

Leave one sin x

$$= \int \sin x \cdot \sin^2 x \sin^2 x \cos^2 x \, dx = \int \sin x (1-\cos^2 x)(1-\cos^2 x)\cos^2 x \, dx$$

$$= \int \cos^6 x \sin x - 2\cos^4 x \sin x \, dx + \cos^2 x \sin x \, dx$$

$u = \cos x \quad du = -\sin x \, dx$

$$= \boxed{-\tfrac{1}{7}\cos^7 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C}$$

Summary

$\int \sin^n x \cos^m x \, dx$ can be done if either m or n is odd, just "save" out one and turn rest into other using $\sin^2 x + \cos^2 x = 1$

Even Powers - last class we saw some reduction formulas.

Ex. $\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$
$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$

Ex. $\int \sin^4 x \, dx = \int \sin^2 x \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$

$$= \int \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2 2x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \int \frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \int \frac{1}{4} + \frac{1}{8}\cos(4x) \, dx$$

$$= \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{x}{4} + \frac{1}{32}\sin(4x) + C$$

<u>Summary</u> $\int \sin^{2m}x\cos^{2n}x$ can be handled with repeated use of Formulas in "3" above.

<u>So Far</u> Should be able to integrate <u>any</u> $\int \sin^n x\cos^m x\,dx$

<u>Recall</u> $\frac{d}{dx}\tan x = \sec^2 x$

<u>Ex</u> $\int \tan^3 x\sec^4 x\,dx$ "save" a $\sec^2 x$

$$= \int \tan^3 x(1+\tan^2 x)\sec^2 x\,dx = \int \tan^3 x\sec^2 x + \tan^5 x\sec^2 x\,dx$$

$$u = \tan x \quad du = \sec^2 x\,dx$$

$$= \boxed{\tfrac{1}{4}\tan^4 x + \tfrac{1}{6}\tan^6 x + C}$$

Works for $\int \tan^n x\sec^{2m}x$

<u>Recall</u> $\frac{d}{dx}\sec x = \sec x\tan x$

<u>Ex</u> $\int \tan^3 x\sec^3 x\,dx$ "save" a $\sec x\tan x$
turn rest into secants

$$= \int (\sec^2 x-1)(\sec^2 x)\sec x\tan x\,dx$$

$$= \int \sec^4 x\sec x\tan x - \sec^2 x\sec x\tan x$$

$$u = \sec x \quad du = \sec x\tan x\,dx$$

$$\boxed{= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C}$$

Works for $\int \tan^{2m+1}x\sec^n x$

Summary $\int \tan^m x \sec^n x \, dx$

1. If n is even, "save" a $\sec^2 x$, turn rest into tangent.

2. If m is odd, save a $\sec x \tan x$, turn $\tan^{m-1} x$ into sec using $\tan^2 x + 1 = \sec^2 x$.

Problem What if m even, n is odd? or $n = 0$?

Tools

- Int by parts
- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Ex $\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \frac{1}{2} \tan^2 x - \ln|\sec x| + C$$

Ex $\int \sec^3 x \, dx$

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x \qquad dv = \sec^2 x \, dx$

$$= \sec x \tan x - \int \tan^2 x \sec x$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x$$

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \tan x + \ln|\sec x + \tan x| \right) + C$$

Problems

1. $\int_0^{\pi/2} \cos^5 x \, dx$

2. $\int \frac{\cos^5 t}{\sqrt{\sin t}} \, dt$

3. $\int \tan^3(2x) \sec^5 2x \, dx$

4. $\int \frac{dx}{\cos x - 1}$

5. $\int t \sec^2(t^2) \tan^4(t^2) \, dt$