Lecture 5

# Chapter 7 Integration Techniques

<u>So far</u>: 1. List of known antiderivatives (see p 452)
2. u-substitution to transform integral into one on our list.

<u>Recall</u>: Product Rule $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

Rearrange: $f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$ now integrate:

$$* \quad \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

(*) is formula for <u>integration by parts</u>

Often written $u = f(x)$ $\quad du = f'(x)\,dx$
$v = g(x)$ $\quad dv = g'(x)\,dx$ so

$$(*) \quad \boxed{\int u\,dv = uv - \int v\,du}$$

<u>How to use (*)</u>

1. Choose $u$ and $dv$ so you have $\int u\,dv$
2. Compute $du$ by differentiating and $v$ by integrating
3. Use (*) and hope $\int v\,du$ is easier!!

<u>Remark</u> Usually choose $u$ so $du$ is simpler
$dv$ so $v$ is not to complicated

**Ex** $\int xe^x dx$ $\quad u = x \quad dv = e^x dx$

$\quad du = 1dx \quad v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$$

**Rmk** Choosing $u = e^x$ $dv = x\,dx$ gives $du = e^x dx$ but $v = x^2/2 + C$ so now must do

$$\int \frac{x^2}{2} e^x dx$$ which is worse!

**Ex** $\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

$u = 2x \quad v = -\cos x$

$du = 2dx \quad dv = \sin x dx$

$u = x^2 \quad dv = \cos x \, dx$

$du = 2x\,dx \quad v = \sin x$

$$= x^2 \sin x - \left(-2x\cos x - \int -2\cos x \, dx\right)$$

$$= x^2 \sin x + 2x\cos x + 2\sin x + C.$$

**Rmk** Take derivative to check:

$x^2\cos x + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x$

**Ex** $\int \ln x \, dx$ $\quad u = \ln x \quad dv = dx$

$\quad du = \frac{1}{x} dx \quad v = x$

$$\int \ln x \, dx = x\ln x - \int 1 dx = x\ln x - x + C$$

$$\boxed{\int \ln x = x\ln x - x + C}$$

Ex. $\int e^x \cos x\, dx \qquad = e^x \sin x - \int e^x \sin x\, dx$

$u = e^x \quad dv = \cos x\, dx \qquad\qquad u = e^x \quad dv = \sin x\, dx$

$du = e^x dx \quad v = \sin x \qquad\qquad du = e^x dx \quad v = -\cos x\, dx$

$$\int e^x \cos x\, dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x\, dx]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$ back where we started!

so $\int e^x \cos x = \frac{1}{2}(e^x \sin x + e^x \cos x)$

Ex. $\int \sin^n x\, dx \qquad u = \sin^{n-1} x \qquad dv = \sin x\, dx$

$du = (n-1)\sin^{n-2} x \cos x \quad v = -\cos x\, dx$

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$(n-1)\int \sin^{n-2} x (1-\sin^2 x)\, dx$$

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

solve for original

$$\boxed{\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx}$$

- Reduction formula, repeated application will allow us to integrate any power of $\sin x$ or $\cos x$, See formulas 73-78 in reference tables.

Ex $\int s 2^s \, ds$ $\qquad u = s \quad dv = 2^s ds$

$\qquad du = ds \quad v = 2^s/\ln 2$

$$= \frac{s 2^s}{\ln 2} - \int \frac{1}{\ln 2} 2^s = \frac{1}{\ln 2} s 2^s - \frac{1}{(\ln 2)^2} 2^s$$

Ex $\int \arctan 4t \, dt$ $\qquad u = \arctan 4t \quad dv = dt$

$\qquad du = 4/1+16t^2 \quad v = t$

$$\int \arctan 4t \, dt = t\arctan 4t - \int \frac{4t}{1+16t^2} dt$$

$$= \boxed{t \arctan 4t - \frac{1}{8} \ln|1+16t^2| + C}$$

Formula # 89: $\int \arctan u \, du = u \arctan u - \frac{1}{2} \ln(1+u^2) + C$

Ex $\displaystyle\int_0^2 \frac{r^3}{\sqrt{4+r^2}} dr$ $\qquad u = r^2 \quad dv = \frac{r}{\sqrt{4+r^2}} dr$

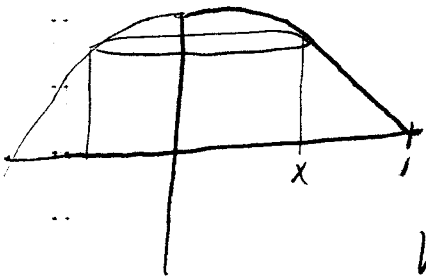
$\qquad du = 2r\,dr \quad v = \sqrt{4+r^2}$

$$= r^2\sqrt{4+r^2} - \int \sqrt{4+r^2}\, 2r\, dr$$

$$= r^2\sqrt{4+r^2} - \frac{2}{3}(4+r^2)^{3/2} \Big|_0^2$$

$$= \left(4\sqrt{8} - \frac{2}{3}(8^{3/2})\right) - \left(0 - 16/3\right)$$

Ex. Find volume rotating $y = \cos\left(\frac{\pi x}{2}\right)$, $y = 0$, $0 \le x \le 1$ about y axis using shells.




height: $\cos(\pi x/2)$

radius $x$

$$V = 2\pi r h \Delta r$$

$$V = \int_0^1 2\pi x \cos(\pi x/2)\, dx \qquad \begin{array}{ll} u = 2\pi x & dv = \cos(\pi x/2)\,dx \\ du = 2\pi\,dx & v = 2/\pi \sin(\pi x/2) \end{array}$$

$$= 4x \sin(\pi x/2) - \int 4 \sin(\pi x/2)\, dx$$

$$= 4x \sin(\pi x/2) + \frac{8}{\pi}\cos(\pi x/2) \Big|_0^1$$

$$= \cancel{(4 + 8/\pi) - (0}$$

$$= (4 + 0) - (0 + 8/\pi) = 4 - 8/\pi$$

Enrichment

Recall: $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

Cor: $\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$

But $\int_0^{\pi/2}\sin^0 x\,dx = \pi/2$, $\int_0^{\pi/2}\sin x\,dx = 1$

so

$$* \quad \int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot \pi/2$$

$$** \quad \int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n+1}$$

#62 Let $I_n = \int_0^{\pi/2}\sin^n x\,dx$. Notice that

1 $I_{2n+2} \le I_{2n+1} \le I_{2n}$

2 $\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$ and $\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$

Thus $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$

But

$$\frac{I_{2n+1}}{I_{2n}} = \underbrace{\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n+1}\cdot\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1}}\cdot\frac{2}{\pi}$$

so must approach $\pi/2$

i.e.

Wallis Formula (1655)

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

converges very slowly.