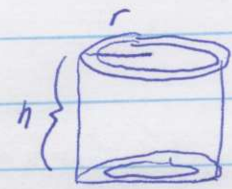
Lecture 4

Review



The volume of a very thin cylindrical shell is

$$V = 2\pi r h \, \Delta r.$$

How to remember? Unroll it!

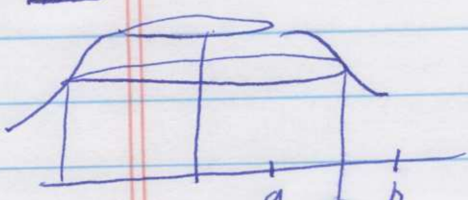
Δr = thickness

Thus we obtained:

Thm Volume obtained by rotating region under $y = f(x) \geq 0$ from $x = a$ to $x = b$ is:

$$V = \int_a^b 2\pi x f(x)\, dx$$

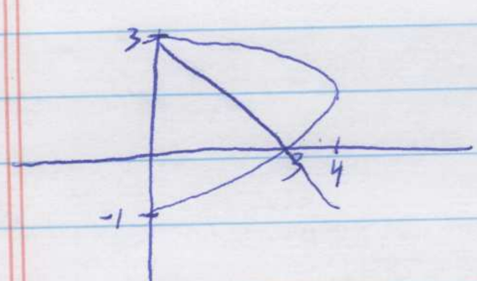
Proof



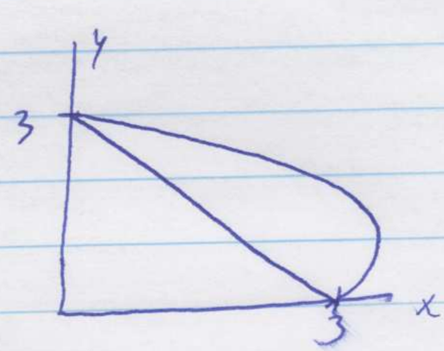
Typical shell has radius x, height $f(x)$

Example

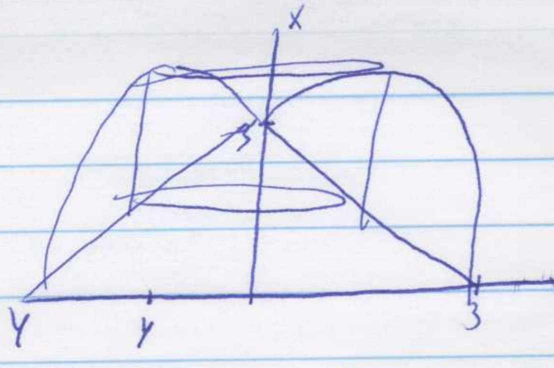
p436 #14 Rotate region bounded by $x + y = 3$, $x = 4 - (y-1)^2$ around x axis and find volume.



$\Longrightarrow$



#14 cont Rotate picture



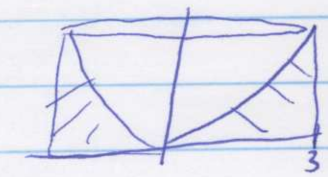
typical shell at y: radius $= y$ height $= 4-(y-1)^2-(3-y)$

$$V = \int_0^3 4-(y-1)^2-(3-y)\,dy = \int_0^3 -y^2+3y\,dy = \frac{-y^3}{3}+\frac{3}{2}y^2\Big|_0^3$$

$$= -9+\frac{27}{2} = \frac{9}{2}$$

Example What region has volume $\int_0^3 2\pi x^5\,dx$?

A: $f(x)=x^4$



under x^4 rotated about y-axis

Rmk $\int_0^3 2\pi x^5\,dx = \frac{\pi}{3}x^6\Big|_0^3 = 243\pi$

By washers: $V = \int_0^{81} \pi\cdot 3^2 - \pi\sqrt[4]{y}^{\,2}\,dy$

$= 9\pi y - \frac{2}{3}\pi y^{3/2}\Big|_0^{81}$

$= 729\pi - 486\pi = 243\pi$

answers agree!

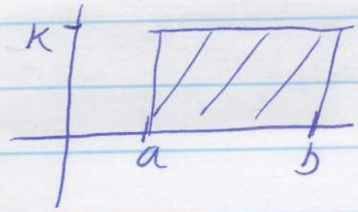
Average Value of a Function

Def The average value of $f(x)$ on $[a,b]$ is:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

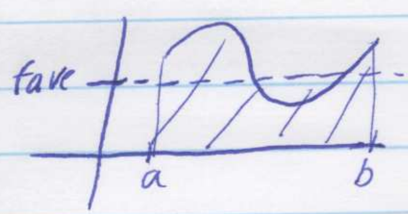
Example / Motivation

Suppose $f(x) = K$ is constant. Then $\int_a^b K\,dx = K\cdot(b-a)$



$$f_{ave} = K = \frac{\text{area}}{b-a}$$

More generally



$$f_{ave} = \frac{\text{area}}{b-a} = \text{avg height}$$

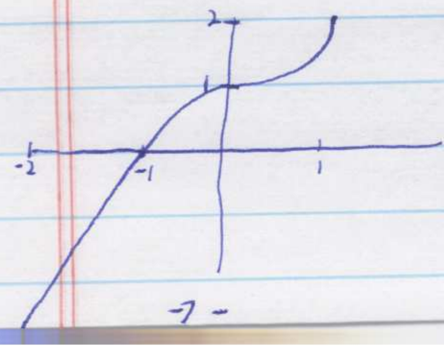
area above $y = f_{ave}$ cancels area below it.

Ex Find average value of $f(x) = 1 + x^3$ on $[-2,1]$

$$f_{ave} = \frac{1}{1-(-2)}\int_{-2}^{1} 1+x^3\,dx = \frac{1}{3}\cdot\left(x+\frac{x^4}{4}\right)\Big|_{-2}^{1}$$

$$= \frac{1}{3}\cdot\left[\frac{5}{4} - (-2+4)\right] = \frac{1}{3}\left(\frac{5}{4}-2\right)$$

$$= -1/4$$



Problem Suppose you drive to Syracuse and average 68 mph. Were you ever going exactly 68 mph?

A: Yes!

Mean Value Thm for integrals

Let f be continuous on $[a,b]$. Then there exists a number c in $[a,b]$ with:

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

i.e.

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

Rmk Speed of your car is (hopefully!) continuous.

Proof Recall mean value thm says $G(x)$ cont, diff'ble on $[a,b]$ then there is c so that

$$\underbrace{G'(c)}_{\text{inst rate of change}} = \underbrace{\frac{G(b)-G(a)}{b-a}}_{\text{avg rate of change}}$$

Let $G(x) = \int_a^x f(t)\,dt$. Then $G'(x) = f(x)$ by FTOC so

There is a c with $G'(c) = \frac{G(b)-G(a)}{b-a}$. Now

$G'(c) = f(c)$, $G(a) = 0$, $G(b) = \int_a^b f(t)\,dt$ //

Ex Let $f(x) = x^2 + x$ on $[1,5]$

a. Find f_{ave}
b. Find c such that $f_{ave} = f(c)$
c. Sketch f and rectangle w/ same area.

Ex Find b so that average value of $f(x) = 2 + 6x - 3x^2$ on $[0,b]$ is equal to 3?

Work

1 Work = Force x distance

units newton-meter = Joule Metric

pound - feet = foot lb English

$1 \text{ ft lb} \approx 1.36$ Joule

2 Force = mass · accel Newton's Second Law.

Ex Find work done lifting 1.2 kg book off floor to desk 1 meter high. Use accel due to gravity is $g = 9.8 m/s^2$.

A: $F = mg = \underbrace{11.76 \, kg \, m/s^2}_{\text{Newton}}$ $W = 11.76 \cdot 1 = 11.76$ joules

Problem Variable force (e.g. magnetic field, gravity on space ships, etc...)

Thm Suppose object moves along x axis from $x = a$ to $x = b$. Suppose $f(x)$ is force acting on object. Then work done is

$$\int_a^b f(x)\,dx$$

force distance

Ex Hooke's Law: $f(x) = kx$ (k = spring constant)

See Ex 3 p 439