Lecture 4

Review

The volume of a very thin cylindrical shell is

\[ V = 2\pi rh \Delta r. \]

How to remember? V=ollw! \( h \) \( \Delta r \)-thickness

Thus we obtained:

Thin volume obtained by rotating region under \( y = f(x) > 0 \) from \( x = a \) to \( x = b \) is:

\[ V = \int_a^b 2\pi x f(x) \, dx \]

Proof:

Typical shell has radius \( x \) height \( f(x) \)

Example:

p. 134 # 14 Rotate region bounded by \( x + y = 3 \), \( x = 4 - (y-1)^2 \) around \( x \) axis and find volume.

\[ \int_{-1}^{3} 2\pi x (3-x) \, dx \]
#14 cont. Rotate picture

Typical shell at $y$: radius = $y$, height = $4 - (y-1)^2 - (3-y)$

\[ V = \int_0^3 2\pi (4-(y-1)^2-(3-y)) \, dy = \int_0^3 (4y^3 + 3y) \, dy = \frac{4y^4}{4} + \frac{3y^2}{2} \bigg|_0^3 = -9 + \frac{33}{2} = \frac{3}{2} \]

Example: What region has volume \( \int_0^3 2\pi x^5 \, dx \)?

A: \( f(x) = x^4 \)

Remark: \( \int_0^3 2\pi x^5 \, dx = \frac{2 \pi}{6} \cdot 3^6 = 243\pi \)

By washes: \( V = \int_0^3 \pi \cdot 3^2 - \pi \cdot \sqrt{3}y^2 \, dy \)

\[ = 9\pi y - \frac{3}{2} \pi y^3 \bigg|_0^3 = 72\pi - 486\pi = 243\pi \]

Answers agree!
Average Value of a Function

Def: The average value of \( f(x) \) on \([a,b] \) is:

\[
\text{ave} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

Example/Motivation

Suppose \( f(x) = K \) is constant. Then

\[
\int_{a}^{b} K \, dx = K \cdot (b-a)
\]

\[
\text{ave} = K = \frac{\text{area}}{b-a}
\]

More generally

\[
\text{ave} = \frac{\text{area}}{b-a} = \text{avg height}
\]

Area above \( y = \text{ave} \) cancels

Area below \( y = \text{ave} \)

Ex: Find average value of \( f(x) = 1 + x^3 \) on \([ -2, 1 ] \)

\[
\text{ave} = \frac{1}{1-(-2)} \int_{-2}^{1} (1 + x^3) \, dx = \frac{1}{3} \cdot \left( x + \frac{x^4}{4} \right)_{-2}^{1}
\]

\[
= \frac{1}{3} \cdot \left[ \frac{5}{4} - (-2 + 1) \right] = \frac{1}{3} \left( \frac{3}{4} - 1 \right) = -\frac{1}{4}
\]
Problem: Suppose you drive to Syracuse and average 68 mph. Were you ever going exactly 68 mph?

A: Yes!

Mean Value Theorem for integrals

Let \( f \) be continuous on \( [a,b] \). Then there exists a number \( c \) in \( [a,b] \) with:

\[
\frac{f(c)}{c} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

i.e.

\[
f(c)(b-a) = \int_a^b f(x) \, dx
\]

Remark: Speed of your car is (hopefully!) continuous.

Proof: Recall mean value theorem says \( f(x) \) cont., diff. on \( [a,b] \), 

there is \( c \) so that

\[
\frac{f'(c)}{c} = \frac{f(b) - f(a)}{b-a} \quad \text{avg rate of change}
\]

Let \( G(x) = \int_a^x f(t) \, dt \). Then \( G'(x) = f(x) \) by FTC so

there is a \( c \) with \( G'(c) = \frac{f(b) - f(a)}{b-a} \) now

\[
G'(c) = f(c), \quad G(a) = 0, \quad G(b) = \frac{1}{a} \int_a^b f(t) \, dt
\]
Ex: Let \( f(x) = \frac{x^2}{x^2 + 1} \) on \([1, 5]\):

a. Find \( f(a) \)

b. Find \( c \) such that \( f(a) = f(c) \)

c. Sketch \( f \) and rectangle with same area.

Ex: Find \( b \) so that average value of \( f(x) = 2 + 6x - 3x^2 \) on \([0, 5]\) is equal to 3.
Work

1. Work = Force \times distance
   
   units: newton \cdot meter = Joule \quad \text{Metric}
   pound \cdot foot = foot \cdot lb \quad \text{English}
   
   1 \text{ ft} \cdot \text{lb} \approx 1.36 \text{ Joule}

2. Force = mass \cdot \text{accel}  \quad \text{Newton's Second Law.}

Ex: Find work done lifting 12 kg book off floor to desk 1 meter high. Use accel due to gravity is \( g = 9.8 \text{ m/s}^2 \).

\[ F = mg = 117.6 \text{ kg} \cdot \text{m/s}^2 \]
\[ W = 117.6 \cdot 1 - 117.6 \text{ Joules} \]

Problem: Variable force (e.g., magnetic field, gravity on space ships, etc.)

Thm: Suppose object moves along x-axis from \( x = a \) to \( x = b \)
Suppose \( f(x) \) is force acting on object. Then

work done is

\[ \int_{a}^{b} f(x) \, dx \]

Ex: Hooke's Law: \( f(x) = kx \) \( (k = \text{spring constant}) \)

See Ex 3 p.439.