Lecture 3

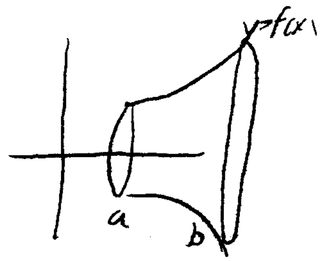
Review: We are using integration to compute volumes:

Thm/Def S solid between $x=a, x=b$. $A(x)=$ area of cross section in plane through x and $\perp$ to x axis Then <u>volume</u> of S is

$$V=\int_a^b A(x)\,dx$$

Ex

Solid of revolution of $y=f(x)\geq 0$, $a\leq x\leq b$ around x-axis



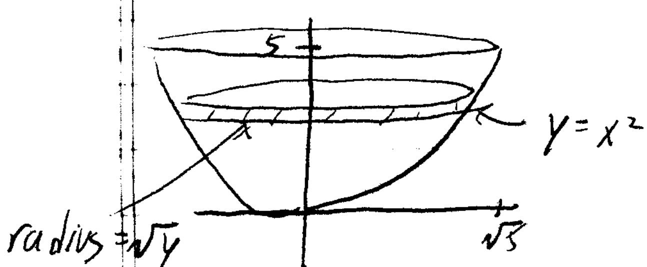
cross section are circles

$A(x)=f(x)^2\cdot\pi$

$$V=\int_a^b \pi f(x)^2\,dx$$

Example

Find the volume of the solid obtained by rotating the region bounded by $y=x^2, y=5, x=0$ around y axis



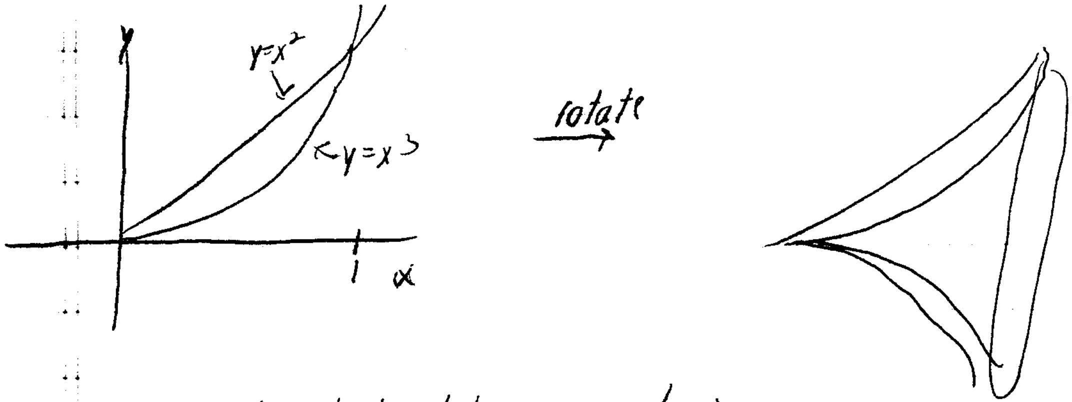
Slice horizontal discs

$0\leq y\leq 5$ $A(y)=(\sqrt{y})^2\cdot\pi$

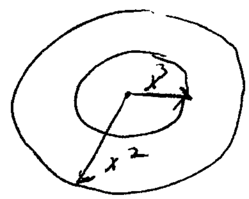
$$V=\int_0^5 \pi y\,dy=\frac{\pi}{2}y^2\Big|_0^5$$

$$=\frac{25\pi}{2}$$

Ex. Take the region enclosed by $y=x^2$ and $y=x^3$ and rotate about the x axis. Find the volume.



Slice looks like "washer"



inner radius x^3
outer radius X

$$A(x) = \pi(x^2)^2 - \pi(x^3)^2 = \pi x^4 - \pi x^6$$

$$\text{Volume} = \int_0^1 \pi x^4 - \pi x^6 \, dx = \frac{\pi}{5}x^5 - \frac{\pi}{7}x^7 \Big|_0^1$$

$$= \pi/5 - \pi/7 = 2\pi/35$$

Sometimes called "washer method," we are slicing into washers of volume $\left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right]\Delta x$

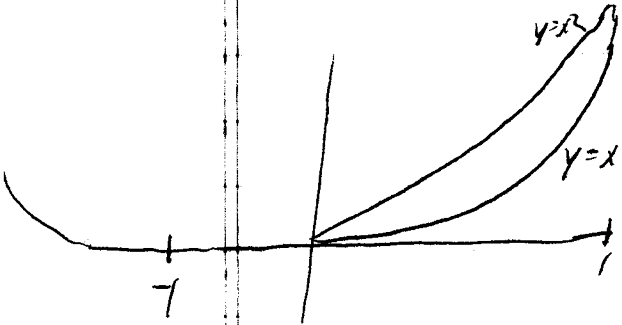
Solids of Revolution

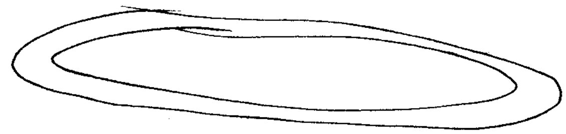
1. Decide which way to slice. $\left[\int_c^d A(y)\,dy \text{ or } \int_a^b A(x)\,dx\right]$

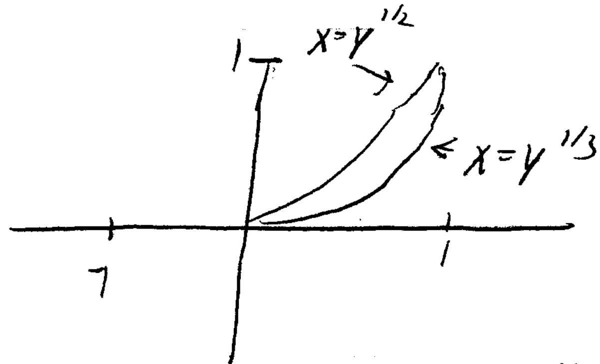
2. Are cross sections discs? Then $A = \pi(\text{radius})^2$

3. Are cross sections washers? Then $A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

Ex Consider region enclosed by $y=x^2$ and $y=x^3$ as above but rotate about line $x=-1$. Find volume.







Slice horizontally.

inner radius $1+\sqrt{y}$ outer radius $1+y^{1/3}$

$$\text{Volume} = \int_0^1 \pi (1+y^{1/3})^2 - (1+y^{1/2})^2 \, dy$$

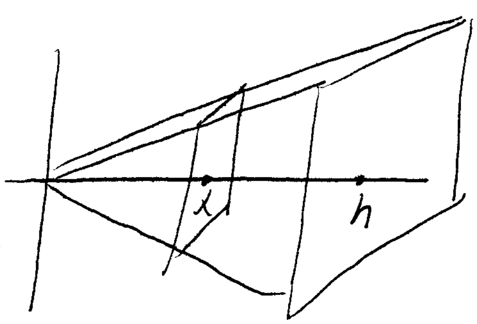
$$= \pi \int_0^1 y^{2/3} + 2y^{1/3} + 1 - 1 - 2y^{1/2} - y \, dy$$

$$= \pi\left(\tfrac{3}{5}y^{5/3} + 3y^{2/3} - \tfrac{4}{3}y^{3/2} - y^2/2\right)\Big|_0^1$$

$$= \pi\left(3/5 + 3 - 4/3 - 1/2\right)$$

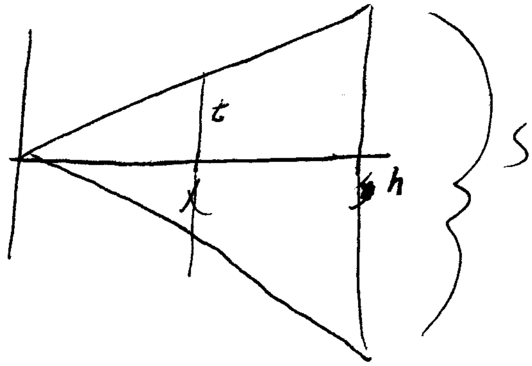
__Solids not of Revolution__

Find volume of pyramid, base square length s, height h.



What is $A(x)$?

See p. 429 for better Picture

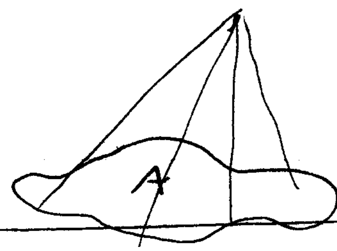


Similar Δ $\frac{x}{h} = \frac{t}{s}$

so $A(x)$ is square side $\frac{xs}{h}$

$$V = \int_0^h \frac{x^2 s^2}{h^2}\,dx = \frac{1}{3}\frac{h^3 s^2}{h^2} = \frac{1}{3} h s^2$$

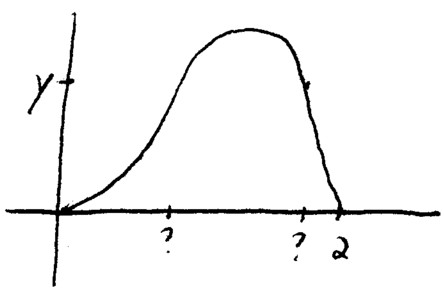
General Formula



$V = \frac{1}{3}$ area base $\cdot$ height

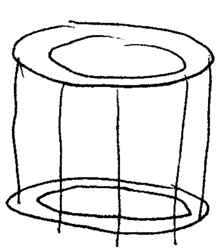
Cylindrical Shell method

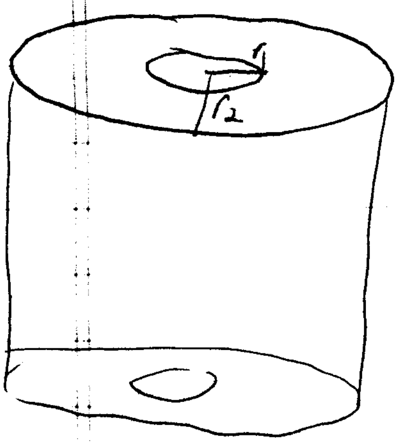
Ex Rotate $y = 2x^2 - x^3$ and $y = 0$ region around y-axis



At given y value, to get inner and outer radii we must solve for x. This is hard, and sometimes impossible.

Goal Slice the volume instead into cylindrical shells:





Inside radius r_1
Outside r_2

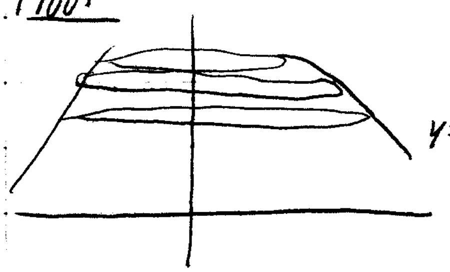
$$V = \pi r_2^2 h - \pi r_1^2 h = \pi h (r_2 - r_1)(r_2 + r_1)$$
$$= 2\pi h \underbrace{\left(\frac{r_2 + r_1}{2}\right)}_{\text{AVG radius}} \underbrace{(r_2 - r_1)}_{\Delta r}$$

So volume $= 2\pi h r \Delta r$

__Thm__ The volume of the solid obtained by rotating region under the curve ~~from~~ $y = f(x)$ from $x = a$ to $x = b$ is

$$V = \int_a^b 2\pi x f(x)\, dx$$

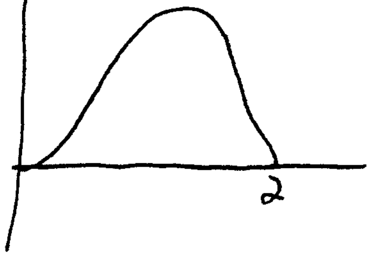
__Proof__



$y = f(x$

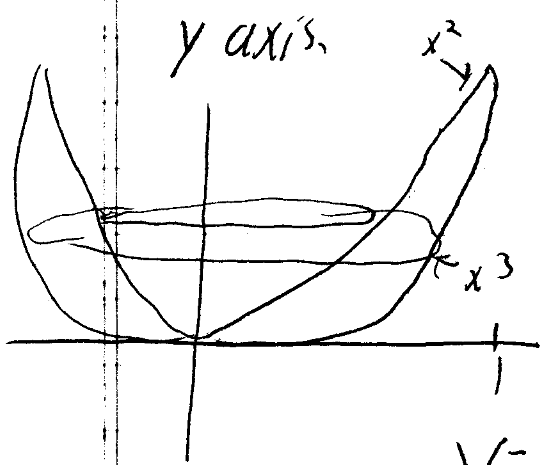
height $h = f(x)$
$r = x$
$\Delta r = dx$ //

__Example__ Problem from before, $y = 2x^2 - x^3$ around y axis



$$V = \int_0^2 2\pi x(2x^2 - x^3)\, dx$$
$$= \int_0^2 4\pi x^3 - 2\pi x^4\, dx$$
$$= \pi x^4 - \frac{2}{5}\pi x^5 \Big|_0^2$$
$$= 16\pi - \frac{64}{5}\pi = 16\pi/5$$

Ex Volume between $y=x^2$ and $y=x^3$ rotated around y axis.



shell has radius x
shell has height x^2-x^3

$$V=\int_0^1 2\pi x\,(x^2-x^3)\,dx$$

$$=\int_0^1 2\pi x^3-2\pi x^4\,dx=\frac{\pi}{2}x^4-\frac{2}{5}\pi x^5\Big|_0^1$$

$$=\pi\left(1/2-2/5\right)$$