

Lecture 24

Review Taylor series of $f(x)$ at $x=a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

• Taylor Remainder Thm For convergence

Ex $f(x) = \frac{1}{1+x}$ Find Maclaurin series from Def.

$$f(x) = \frac{1}{1+x} \quad f'(x) = \frac{-1}{(1+x)^2} \quad f''(x) = \frac{2}{(1+x)^3} \quad f'''(x) = \frac{-3 \cdot 2}{(1+x)^4} \quad f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{(1+x)^5}$$

$$f(0) = 1 \quad f'(0) = -1 \quad f''(0) = 2 \quad f'''(0) = -3! \quad f^{(4)}(0) = (-1)^4 4!$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n = 1 - x + x^2 - x^3 + x^4 \dots$$

Mult and Division Taylor series multiply and divide like polynomials

Ex $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\begin{array}{r} e^x \cos x = \\ \begin{array}{r} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \end{array} \\ \hline \begin{array}{r} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \\ \quad - \frac{x^2}{2} \quad - \frac{x^3}{2} \quad - \frac{x^4}{4} \quad - \frac{x^5}{12} \\ \quad \quad \quad \frac{x^4}{24} \quad + \frac{x^5}{24} \dots \end{array} \\ \hline 1 + x - \frac{x^3}{3} \dots \end{array}$$

Ex

$$\tan x = \frac{\sin x}{\cos x}$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \dots \left| \begin{array}{l} x + \frac{1}{3}x^3 \\ \hline x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \\ \hline x - \frac{x^3}{2} + \frac{1}{24}x^5 \\ \hline \frac{1}{3}x^3 - \frac{1}{24}x^5 \end{array} \right.$$

etc...

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

Fact • Multiplying two Taylor Series w/ $ROC = R$ gives new ones with ROC at least R

• Dividing by $\sum b_n x^n$ w/ $b_0 \neq 0$ gives Taylor series converging for some positive ROC .

Applications

Recall Tangent line to $y=f(x)$ at $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a) \quad \text{This is 1st Taylor polynomial!!}$$

Thm $T_n(x)$ gives the best degree n approximation to $f(x)$ in a certain precise sense.

Ex $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ How big is error if $-.3 \leq x \leq .3$?
Approximate $\sin(8^\circ)$ this way.

A $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ so by A.S. error estimate

$$|\text{Error}| \leq \frac{.3^7}{7!} = .00000004339$$

$$\sin(8^\circ) = \sin\left(\frac{8\pi}{180}\right) = \sin\left(\frac{2\pi}{45}\right) \approx \frac{2\pi}{45} - \frac{(2\pi/45)^3}{6} + \frac{(2\pi/45)^5}{120}$$

$$\approx .13962634 - .0004536812 + .000004421$$

$$= .13917310$$

$$\sin(8^\circ) \text{ actually} \approx .139173100$$

Remark Such simplifications often used in physics

$$\text{Indeed for } x \text{ near } 0, \sin(x) \approx x, \quad |\text{error}| \leq \frac{x^3}{6}$$

Problems

p. 755 # 9, 13, 26

p. 747 # 51, 36