Lecture 23

Review

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has R.O.C $R > 0$. Then

$$f'(x) = \sum_{n=0}^{\infty} n \cdot c_n (x-a)^{n-1} \quad \text{and} \quad \int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$

also have R.O.C. R.

Thus starting w/ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ plus subst, diff, $\int dx$ we can get power series for

$\frac{x}{1+x^2}$, $\ln|1+x|$, etc... This is somewhat ad hoc!

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$ for $|x-a| < R$.

Notice: $\boxed{c_0 = f(a)}$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

$$\boxed{c_1 = f'(a)}$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \dots$$

$$2c_2 = f''(a) \qquad \boxed{c_2 = \frac{f''(a)}{2}}$$

$$f'''(x) = 3\cdot 2\cdot 1c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \dots$$

$$3\cdot 2\cdot 1c_3 = f'''(a) \qquad c_3 = \frac{f'''(a)}{3!} \qquad \text{ETC...} \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

Thm. If f(x) has a power series expansion at a, i.e. if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R \text{ with } R > 0,$$

then $c_n = \dfrac{f^{(n)}(a)}{n!}$

Def. The Taylor series of f(x) centered at $x = a$ is:

$$f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Def. When $a=0$ the Taylor series is called Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Situation If f(x) has a power series it is given by the Taylor series but we need to know for which x this holds

Ex $f(x) = e^x$ $f'(x) = e^x \ldots f^{(n)}(x) = e^x$; so $f^{(n)}(0) = 1$ so

Maclaurin Series of e^x $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Does this converge to e^x for any x?

Def Given $f(x)$, the <u>n^{th} Taylor polynomial of $f(x)$ at a</u> is

$$T_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>Ex</u> $f(x) = e^x$ at $x = 0$.

$$T_1(x) = 1 + x \quad T_2(x) = 1 + x + \frac{x^2}{2!} \quad \ldots$$

Def The n^{th} remainder is $R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots$

<u>Thm</u>

Suppose Taylor series is $f(x) = T_n(x) + R_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ then $f(x)$ is equal to its Taylor series on $|x-a| < R$.

<u>Taylor's Inequality</u> Suppose $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$

Then $\quad |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

* Use to prove $\lim_{n\to\infty} |R_n(x)| \leq 0$.

Ex

Claim $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

Proof $f^{(n)}(x) = e^x$. Choose $M = e^d$ in Taylor's inequality. Then

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d.$$

Useful Fact

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } n.$$

Thus $\lim_{n\to\infty} |R_n(x)| = 0$ for any x by Sqz Thm. To

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad -\infty < x < \infty}$$

Ex Find Maclaurin Series for $f(x) = \sin x$ and prove it represents $\sin x$ for all x.

Solution $f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad f'''(x) = -\cos x$ $\quad f^{(4)}(x) = \sin x$

$f(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$ $\quad f'''(0) = -1$ $\quad f^{(4)}(0) = 0 \ldots$

Maclaurin series: $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots$

Notice $M = 1$ works for any d in Taylor's Ineq so

$$|R_n(x)| \le \frac{x^{n+1}}{(n+1)!} \quad \text{so} \quad \lim_{n\to\infty} |R_n(x)| = 0 \text{ any } x.$$

Thus

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Also

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Observe: $(\sin x)' = \cos x$, etc..

Ex Compute Taylor series for $f(x) = \ln x$ at $x=1$

$f(x) = \ln x \qquad f(1) = 0$
$f'(x) = 1/x \qquad f'(1) = 1$
$f''(x) = -1/x^2 \qquad f''(1) = -1$
$f'''(x) = 2/x^3 \qquad f'''(1) = 2/1$
$f^{(4)}(x) = -6/x^4 \qquad f^{(4)}(1) = -6$
$f^{(5)}(x) = +24/x^5 \qquad f^{(5)}(1) = 24$

so $f^{(n)}(1) = (-1)^{n-1} \cdot (n-1)!$

Taylor series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$= (x-1) - \frac{(x-1)}{2} + \frac{(x-1)}{3} - \frac{(x-1)}{4} \cdots$$

Notice it it $x>0$ then $\exists$ an M for Taylor ineq.
Thus $R=1$

$$\ln(x) = x-1 - \frac{x-1}{2} + \frac{x-1}{3} - \frac{x-1}{4} \cdots \qquad 0 < x \leq 2$$

$$\ln(2) = 1 - 1/2 + 1/3 - 1/4 - \cdots$$

Important Taylor Series To Know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots \qquad R = 1$$