Review

Def $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is <u>power series centered at a</u>.

Given such a power series, we are interested in knowing which x it converges for.

Ex $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \cdots$ Ratio test: $\lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} nx$
$= \infty$ if $x \neq 0$

This series converges only at $x=0$

Ex $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges for all x in $(-\infty, \infty)$.

Thm Given $\sum_{n=0}^{\infty} c_n(x-a)^n$, one of 3 possibilities occurs:

1. Converges only at $x=a$. $(R=0)$
2. Converges for all x. $(R=\infty)$
3. Converges for $|x-a|<R$ and diverges for $|x-a|>R$.

- R is radius of convergence
- At endpoints $a-R$, $a+R$ anything can happen. Thus possible <u>intervals of convergence</u> are $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$

Ex $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = x-2 + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-3)^3 + \cdots$ Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n}|x-2|$
converges for $|x-2|<1$.

<u>Endpoints</u> $x=3$ $1 + 1/2 + 1/3 + 1/4 + \cdots$ diverges

$R=1$

$x=1$ $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$ converges
Interval of conv. $[1,3)$.

Ex Find R.O.C. and I.O.C.

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} \qquad \text{Ratio test: } \lim_{n\to\infty} \frac{\frac{x^{n+1}}{4^{n+1}\ln(n+1)}}{\frac{x^n}{4^n \ln n}}$$

$$= \lim_{n\to\infty} \frac{x \ln n}{4 \ln(n+1)} \overset{LHR}{=} \lim_{n\to\infty} \frac{x \cdot 1/n}{4 \cdot 1/(n+1)} = x/4$$

Thus $-1 < x/4 < 1 \Rightarrow -4 < x < 4$ so R.O.C. $= 4$.

Endpoints $x = 4$ $\quad \sum_{n=2}^{\infty} (-1)^n \cdot 1/\ln n$ converges by AST

$x = -4$ $\quad \sum_{n=2}^{\infty} 1/\ln n$ diverges by comparison to $1/n$.

I.O.C. $(-4, 4]$

Ex. Find R.O.C. and I.O.C.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n^3} = x - \frac{1}{8}x^2 + \frac{1}{27}x^3 - \frac{1}{64}x^4 - \ldots$$

Ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{x}{(n+1)^3}}{\frac{x}{n^3}} = \lim_{n\to\infty} x \cdot \frac{n^3}{(n+1)^3} = x.$

So converges for $-1 < x < 1$. $\boxed{\text{R.O.C.} = 1}$

Endpoints $x = 1$ $\quad \sum_{n=1}^{\infty} (-1)^{n-1} 1/n^3$ conv.

$x = -1$ $\quad \sum_{n=1}^{\infty} 1/n^3$ conv.

So I.O.C. $= [-1, 1]$

Ex Bessel function $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!\, 2^{2n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}}}{\dfrac{x^{2n+1}}{n!(n+1)!\,2^{2n+1}}} = \lim_{n\to\infty} \frac{x^2}{2^2\cdot(n+2)(n+1)} = 0 \text{ for all } x.$$

I.O.C $= (-\infty, \infty)$ ROC $= \infty$

Solution to

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0$$

heat conduction
electromagnetic waves

Problem

Given function, like $f(x) = e^x$, can we find a power series that converges to $f(x)$ on some $(a-R, a+R)$??

One way - start with what we know

Ex $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1<x<1$ (last class)

So $\frac{1}{1+2x} = 1 - 2x + (-2x)^2 + (-2x)^3 + (-2x)^4 + \dots \quad -1 < -2x < 1$

$= 1 - 2x + 4x^2 - 8x^3 + 16x^4 \dots \quad -\frac{1}{2} < x < \frac{1}{2}$

$= \sum_{n=0}^{\infty} (-1)^{n-1} 2^n x^n$

Ex $\frac{x^3}{1+2x} = x^3 - 2x^4 + 4x^5 - 8x^6 + \dots = \sum_{n=0}^{\infty} (-1)^{n-1} 2^n x^{n+3}$

<u>Question</u> $\frac{1}{1-x} = 1+x+x^2+x^3+\dots \quad -1<x<1$ Can we take integral or derivative term by term?

$$= \sum_{n=0}^{\infty} x^n$$

i.e. take $\frac{d}{dx}$:

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots = \sum_{n=0}^{\infty}(n+1)x^n$$ is this legit?

or take $\int - \, dx$

$$-\ln|1-x| = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots = \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$$

Answer: Yes!!

<u>Thm</u> Suppose $\sum_{n=0}^{\infty} c_n(x-a)^n$ has R.O.C. $R>0$. Then:

1. $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \cdot \frac{1}{n+1}(x-a)^{n+1}$

And power series for $f'(x)$ and $\int f(x)\,dx$ also have R.O.C. R.

<u>Rmk</u> Convergence on endpoints $a-R$, $a+R$ may change between $f(x)$, $f'(x)$, $\int f(x)\,dx$, i.e. <u>interval</u> of convergence ~~is~~ may not be same

Ex $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ so $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} x^{2n}(-1)^n$ Thus

$$\tan^{-1}x = \int \frac{1}{1-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} x^{2n+1} + C$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + C$$ Plug in $x=0$ to get $C=0$

$$\boxed{\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \dots \quad \text{for } -1<x<1}$$

Ex Find power series and interval of convergence

$$f(x) = \frac{3}{x^2-x-2}, \qquad \frac{3}{x^2-x-2} = \frac{A}{(x-2)} + \frac{B}{x+1}$$

$$3 = A(x-1) + B(x-2)$$

$B=-3 \quad A=3$

$$f(x) = \frac{3}{x-2} + \frac{-3}{x+1}$$

$$= \frac{-3/2}{1-\frac{2}{3}x} \qquad \frac{-3}{1- -x} =$$

$$\frac{-3}{2}\sum_{n=0}^{\infty}\left(\frac{-2}{3}\right)^n x^n \qquad -3\sum_{n=0}^{\infty}(-x)^n \quad -1\le x\le 1$$

for

$-1 \le -2/3\, x \le 1$

$-3/2 \le x \le$

$$= \sum_{n=0}^{\infty}\left(\frac{-3}{2}\cdot\left(\frac{-2}{3}\right)^n - 3\cdot(-1)^n\right)x^n$$

$-1<x<1$

Ex Use a power series to approximate $\int_0^{.3} \frac{x^2}{1+x^4}\,dx$ w/in 6 dec places

$$\frac{1}{1-x} = 1+x+x^2+x^3 \cdots$$

$$\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = 1 - x^4 + x^8 - x^{12} + x^{16} \cdots$$

$$\frac{x^3}{1+x^4} = x^2 - x^6 + x^{10} - x^{14} + x^{18}$$

$$\int \frac{x^2}{1+x^4}\,dx = \frac{x^3}{3} - \frac{x^7}{7} + \frac{x^{11}}{11} - \frac{x^{15}}{15} + \frac{x^{19}}{19}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n-1}}{4n-1}$$

$$\int_0^{.3} \frac{x^2}{1+x^4} = \frac{(.3)^3}{3} - \frac{(.3)^7}{7} + \frac{(.3)^{11}}{11} - \frac{(.3)^{15}}{15} + \frac{(.3)^{19}}{19}$$

converges by AST.

$$\frac{.3^{11}}{11} = .000000161042$$

$$\frac{.3^3}{3} - \frac{.3^7}{7} = .008968757$$

$+ =$

$.008968918$

Answer $\boxed{.008969}$