Review

A.S.T. If $a_1 \geq a_2 \geq a_3 \cdots > 0$ and $\lim_{n\to\infty} a_n = 0$ then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 \ldots \text{ converges.}$$

Remainder Estimate. Suppose $\sum (-1)^{n-1} a_n = S$ converges by A.S.T. Then

$|R_n| = |S - S_n| \leq a_{n+1}$, i.e. remainder in absolute value is $\leq$ next term.

Example $\sum_{n=2}^{\infty} (-1)^{n-1} \cdot 1/\ln n = -1/\ln 2 + 1/\ln 3 - 1/\ln 4 + 1/\ln 5 \cdots$ converges

and if $\sum_{n=2}^{\infty} (-1)^{n-1} 1/\ln n = S$ then $|S - (-1/\ln 2 + 1/\ln 3 - \ldots + 1/\ln 11)| \leq 1/\ln 12$

Def Absolute convergence means $\sum |a_n|$ converges

Ratio Test Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. Then

$$\begin{cases} L > 1 & \text{means } \sum a_n \text{ diverges} \\ L = 1 & \text{no info} \\ L < 1 & \text{means } \sum a_n \text{ converges absolutely.} \end{cases}$$

Proof Comparison to geometric series

Root Test Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. Then same conclusion as ratio test.

~~Ex~~ ~~p~~
Ex $\sum_{n=1}^{\infty} \left(\frac{n^2+3}{6n^2+1}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{n^2+3}{6n^2+1}\right| = 1/6$

converges absolutely
by root test.

Strategy For Deciding if $\sum a_n$ converges (and if so is it absolute or conditional?)

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1. Know p-series ($\sum 1/n^p$ converges iff $p>1$) and geometric series, $\sum ar^{n-1}$ converges iff $|r|<1$

2. Unless $\lim_{n\to\infty} a_n = 0$, the series automatically diverges.

3. If close to p-series or geom series, perhaps comparison or Limit Comparison Test.
   - only apply to positive series
   - however $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges so may be used to show absolute convergence.

4. If series is alternating then A.S.T. obvious possibility

5. $n!$, a^n's suggest ratio test

6. $a_n = (b_n)^n$ suggest root test.

7. If $a_n = f(n)$ and $\int_1^{\infty} f(x)$ is doable, think integral test.

Examples

1. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ Recall $\lim_{h\to 0}\frac{\sin(h)}{h}=1$

$$= \sum_{n=1}^{\infty} \frac{\frac{\sin(1/n)}{1/n}}{\frac{\sqrt{n}}{1/n}}$$

so compare to $1/n^{3/2} = 1/n\sqrt{n}$

$$\lim_{n\to\infty} \frac{\frac{\sin(1/n)}{\sqrt{n}}}{\frac{1}{n\sqrt{n}}} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = 1$$

Thus series converges by L.C.T.

2. ~~lim~~ $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n!}{n^4} = \infty$$ Thus DIVERGES

3. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ now $\frac{1}{n+n\cos^2 n} \ge \frac{1}{2n}$ so DIVERGES by comparison test.

4. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$ $\lim_{n\to\infty}\frac{\sqrt[n+1]{2}-1}{\sqrt[n]{2}-1} = ? \lim_{x\to\infty}\frac{2^{1/x+1}-1}{2^{1/x}-1}$

$$y = \frac{\sqrt[n+1]{2}-1}{\sqrt[n]{2}-1} =$$

5. $\sum_{n=1}^{\infty} n!/e^{n^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}}$$

$$= \frac{(n+1)e^{n^2}}{e^{(n+1)^2}} = \frac{(n+1)e^{n^2}}{e^{n^2+2n+1}}$$

$$= \frac{n+1}{e^{2n+1}}$$

and $\lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0$ so converges by ratio test.

6. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ Notice $k \ln|k| < k\sqrt{k}$ so

$0 \le \frac{k \ln k}{(k+1)^3} \le \frac{k^{1.5}}{k^{3+\ldots}}$ converges comparison test

7. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n+2)}$

8. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

Power series

Def A power series is a series of the form:

$$c_0 + c_1 x + c_2 x^2 + \dots = \sum_{n=0}^{\infty} c_n x^n, \quad x \text{ is variable, } c_i \text{ coefs}$$

Ex 1. $1 + x + x^2 + x^3 + \dots = \sum x^n$

2. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3. $c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n (x-a)^n$

power series centered at a

Problem

Is $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$ a well-defined function?

Answer Yes, precisely for values of x that make series converge!

Example

For what x values is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent?

A: Ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$ for all x!

FACT $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ for all x.

Ex $1 + x + 2x^2 + 3x^3 + 4x^4$ clearly diverges if $|x| \geq 1$.

What about $-1 < x < 1$?