Lecture 20

Review of Tests So Far:

Earlier: 1. If $\sum |a_n|$ converges then $\sum a_n$ converges

2. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

Integral Test

Setting: $f(x)$ continuous, eventually decreasing on $[1,\infty)$ and $a_n = f(n)$.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Rmk If sum starts not at $n=1$, just adjust, e.g. $\sum_{n=4}^{\infty} \frac{1}{(n-2)^2}$ and $\int_4^{\infty} \frac{1}{(x-2)^2}\,dx$.

Remainder Estimate Suppose $a_n = f(n)$ as above and $\sum_{n=1}^{\infty} a_n = S$ converges

Let

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots \quad n^{th} \text{ remainder. Then:}$$

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

Comparison Test Setting: $a_n, b_n \ge 0$.

1. If $a_n \le b_n$ eventually and $\sum b_n$ converges then $\sum a_n$ converges

2. If $a_n \le b_n$ eventually and $\sum a_n$ diverges then $\sum b_n$ diverges

Limit Comparison setting $a_n, b_n \ge 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ $(c \neq \infty)$

then either

both $\sum a_n$, $\sum b_n$ converge or both diverge.

Ex

1. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ 2. $\sum_{n=1}^{\infty} \frac{\sqrt{n-2}}{n^3+n+1}$ 3. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

Estimate error w/ 1^{st} 10 terms

Alternating Series

Def A series is <u>alternating</u> if terms are alternately positive and negative.

Ex 1. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$ <u>alternating harmonic series</u>

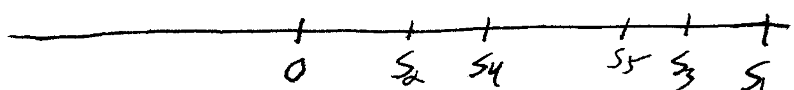
2. Anytime $\sum a_n$ is positive series then $\sum(-1)^n a_n$, $\sum(-1)^{n+1}a_n$ are alternating.

Thm (Alternating Series Test)
Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1-b_2+b_3-\ldots$ is alternating (so $b_n > 0$) and

1. $b_{n+1} \le b_n$
2. $\lim_{n\to\infty} b_n = 0$. Then it converges.

Ex $1-1/2+1/3-1/4+1/5-1/6$ converges since $1/(n+1) \le 1/n$ and $\lim_{n\to\infty}\frac{1}{n}=0$.

Proof



Each partial sum "bounces" back, not as far as before. Thus

$S_2 \le S_4 \le S_6 \ldots$
$S_1 \ge S_3 \ge S_5 \ldots$ both monotonic, so $\lim_{n\to\infty} S_{2n}$, $\lim_{n\to\infty} S_{2n+1}$ exist
However, since $a_n \to 0$ both are $=$.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}} = -\frac{1}{1^{3/4}} + \frac{1}{2^{3/4}} - 1/3^{3/4} \cdots$ converges

Ex. $1/\sqrt{2} - 1/\sqrt{3} + 1/\sqrt{4} - 1/\sqrt{5} \cdots$ converges.

Ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$ diverges. (A.S.T. does not apply)

Observation Notice in our proof that the actual sum always lies between s_n and s_{n+1}.

Alt Series Estimation Suppose $0 \le b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$. (i.e. suppose AST applies)

Let $s = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Then

$$|s - s_n| = R_n \le b_{n+1}.$$

i.e. |remainder| $\le$ next term.

Ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + 1/7 - 1/8 + 1/9 = s_9$

$1 - 1/2 + 1/3 - 1/4 + \cdots\cdots = \ln 2$. Thus s_9 is an overestimate and

$$|s - s_9| = |\ln 2 - s_9| \le 1/10.$$

Ex Find $\displaystyle\sum_{n=0}^{\infty} (-1)^n/n!$ correct to 3 decimal places

WARNING THIS ERROR ESTIMATE APPLIES ONLY IF A.S.T. applies other alt series may still converge.

Absolute Convergence

DEF $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Examples

1. If all $a_n \geq 0$ then absolute convergence is same as convergence.

2. $\sum_{n=1}^{\infty} (-1)^n / n^2$ converges absolutely

3. $1 + 1/8 - 1/3^3 + 1/4^3 + 1/5^3 - 1/6^3 + 1/7^3 + 1/8^3 - 1/9^3 + \ldots$
   converges absolutely.

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot 1/n = 1 - 1/2 + 1/3 - 1/4 + \ldots$ converges, but not absolutely.

Def A series $\sum a_n$ is conditionally convergent if it is convergent but not absolutely.

Thm If $\sum a_n$ is absolutely convergent then it is convergent.

Proof $0 \leq a_n + |a_n| \leq 2|a_n|$ Since $\sum |a_n|$ converges then $\sum 2|a_n|$ does

so $\sum a_n + |a_n|$ converges by Comparison
$- \sum |a_n|$ converges by Assumption

Thus $\sum a_n$ converges

Ex Prove that $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^{3/2}}$ converges.

Last Test..

Thm (Ratio Test)

1. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Then $\sum a_n$ is absolutely convergent

2. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Then $\sum a_n$ diverges

3. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ Then Ratio Test gives no info

Proof Basically comparison w/ geometric series

Ex. Does $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^5}{2^n}$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^5}{2^{n+1}}}{\frac{n^5}{2^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^5}{2n^5} = 1/2$$

so converges absolutely.

Ex. Is $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ convergent?

Rmk. Ratio Test works well with things like $n!$, 3^n, etc..

Thm. 1. Suppose $\sum a_n$ converges absolutely, then any rearrangement has same sum.

2. Suppose $\sum a_n$ converges conditionally. Then rearranging, we can make it converge to anything!!!